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THE RATE OF INTEREST AND THE VALUE OF CAPITAL WITH UNLIMITED SUPPLIES OF LABOUR

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Introduction

This essay investigates the choice of technique (labor:capital ratio) and the choice of the rate of saving as joint decisions linked by the following mechanism: (1) the supply of labor is always infinitely elastic at an exogenously determined wage rate; (2) all wage-income is consumed; and (3) the marginal disutility of labor as well as its productivity unassisted by capital are nil.[1] The principal conclusions of this investigation are, first, that for the optimal technique and saving rate, the marginal productivity of labor in the capitalistic sector lies between the wage rate and zero. Second, and more important, neither the private nor the social rate of return (or marginal productivity) of capital is equal to the subjective rate of interest defined by the marginal premium on present over future consumption implicit in the economy's social welfare function; optimally, the subjective rate of interest is equal rather to the physical marginal productivity of capital. The difference between the social and physical productivity of capital is the difference between a mutatis mutandis and a ceteris paribus change. The social return measures the extra output from an extra unit of output of capital if employment increases sufficiently to maintain the socially optimal labor:capital ratio, which is, of course, the correct employment strategy under the assumptions of this essay. The physical return to capital measures the extra output under the assumption that employment does not change with the addition of a unit of capital.

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[1] These assumptions represent a theoretical simplification of the framework of growth going back at least to Karl Marx. For a modern discussion, see W. Arthur Lewis, "Economic Development With Unlimited Supplies of Labor," Manchester School, 22, 1954, 139, and "Unlimited Labor : Further Notes," Manchester School, 26, 1958, 1. See also Lewis's book, The Theory of Economic Growth, George Allen & Unwin, 1955.

The implication of this second conclusion for investment planning will be discussed later, but the extreme nature of our assumptions about the availability and behavior of labor compel at least cursory attention at the outset to the relevance of these assumptions. Stated baldly they are far from realistic, especially the assumption of perpetually unlimited supplies of labor at a fixed wage. But the germ of truth that makes the assumption of unlimited supplies of labor worth exploring is that in many underdeveloped economies unemployment and underemployment is large, and the wage rate of unskilled labor is well in excess of its opportunity cost measured in terms of either marginal disutility or of alternative product foregone. And worse, in many countries the creation of employment opportunities hardly keeps pace with the growth of the labor force. In India, for example, the relative as well as absolute amount of unemployment has apparently increased since Independence, despite fifteen years of planned economic development.[1] This is not a state of affairs that will continue in perpetuity, one hopes, but certainly the wage rate will exceed the opportunity cost of labor for some time to come, and India is not unique in this respect.

The assumption that workers consume their entire wage-income may seem inappropriate in a model which attempts to simulate the choice of saving rate as well as technique. With unlimited supplies of labor, surely the labor:capital ratio should be increased until the marginal productivity of labor falls to zero, and the

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1 The following estimates are taken from V.R.K. Tilak, "Unemployment Statistics in India" Economic Weekly 27, 1965, p. 27.

Year	At the beginning of	Number unemployed (millions)
1951	First Plan	Not available
1956	Second Plan	5.3
1961	Third Plan	{9.0 (original) 8.0 (revised)}
1966	Fourth Plan	12.0 (anticipated)

The compound growth rate of unemployment is over 8%, as compared with a growth rate of population of the order of 2%. Underemployment is, of course, more difficult to estimate.

consumption of workers should be a separate issue. Even if workers cannot be induced to save voluntarily, it ought to be possible to force savings out of wages through a combination of taxation and reduction of real wages by means of inflation.[1] However, governments are in general severely restricted in their ability to control the rate of consumption out of wage income. In decentralized, pluralistic societies, organized labor -- along with other interested groups -- can be expected to resist the taxation and inflation which would be required to force savings from wage income. And this resistance is likely to be effective, for the political advantages of increasing employment are relatively few. The unemployed after all, are a minority of the labor force even in the most labor-surplus economies, so that even if man for man the unemployed were equally powerful politically as the employed, the sheer weight of numbers would make the interest of the employed in low taxes and price stability carry the day against the interest of the unemployed in expansion of the volume of investment and hence employment.[2]

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1 Such an assumption as this is implicit in Francis Bator's willingness to assume that the choice of a rate of saving can be made independently of the distribution of income. See "On Capital Productivity, Input Allocation, and Growth," Quarterly Journal of Economics, 71, 1957, p. 98. Bator admits the logical possibility of a link between income distribution and savings (p. 103), but does not appear to take the problem such a link would pose very seriously.

2 And the irony of this conflict of interest is that the more successful a government is in increasing the volume of savings and employment by taxation or inflation, the more difficulty it encounters. For the very people who are moved from the ranks of the unemployed to the employed, who might be expected to be the most vocal supporters of the taxation or inflation that created their jobs, now identify their interests with those who were already employed and hence lose from taxation and inflation.

Even in more highly planned and centralized economies, the latitude of the government to increase savings and investment by decreasing the consumption per employed worker is limited. Joseph Pajestka indicates that the attempts of the Polish Government to do just this in the decade following the defeat of Nazi Germany "placed heavy burdens on certain social groups and brought in their wake the well-known social-political reactions and dispositions which resulted in checking further economic development."[1]

The Polish experience points up that it is consumption per worker rather than per capita that is at issue. More intensive use of existing capital goods makes it possible to increase total (and thus per capita) consumption and investment, at least to the point that the marginal productivity of labor falls to zero. But to increase the labor:capital ratio beyond the point where the marginal productivity of labor falls to the level of the wage necessitates either a fall in real consumption per worker or a fall in the rate of profit per unit of capital and hence in the rate of investment and growth. That consumption per worker rather than consumption per capita should be the politically sensitive magnitude is perhaps not so surprising after all. A society need not be Calvinistic for there to exist differences in expectations and aspirations between the employed and the unemployed. Individuals may become inured to chronic underemployment or unemployment, but like individuals in possession of jobs may feel legitimately entitled to some minimum level of consumption in return for a day's work and exercise all the political power at their command to resist taxation or inflation which might deprive them of their accustomed standard.

1 J. Pajestka, "Some Problems of Economic Development Planning," p. 323, in O. Lange, (ed.), Problems of Political Economy of Socialism, People's Publishing House, New Delhi, 1962.

The preceding discussion is not intended to suggest a belief on my part in the absolute realism of the assumptions that underlie the model analyzed in this essay. The ingredients of theoretical models generally represent an extreme simplification of the actual environment of economic decisions, and the present case is no exception. Nevertheless, the model examined in subsequent sections of this essay captures sufficiently the distinctive features of a large number of countries in Asia and elsewhere to make it worthwhile to explore its implications for development policy.

Technique and Saving Divorced

To provide a basis of comparison, it may be a good idea first to set out the relevant results under the assumption that the government is able, by one means or another, to achieve any desired rate of savings regardless of the labor:capital ratio chosen. Thus the choice of technique can be divorced from the savings discussion. Given unlimited supplies of labor and our assumption that both the disutility of labor and labor-productivity unassisted by capital are zero, we may suppose that the labor intensity is chosen to maximize the output:capital ratio regardless of the level of the wage rate. In other words, labor intensity is increased until the marginal productivity of labor in the capitalistic sector is driven to zero. So much for the choice of technique.

Following Ramsey,[1] the optimal savings program is defined as one which minimizes the integral over the interval $[0,\infty)$ of the difference between "bliss" (the least upper bound on instantaneous utility) and the utility actually achieved. If we

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[1] F. P. Ramsey, "A Mathematical Theory of Saving," *Economic Journal*, *38*, 1928, 543.

denote consumption at time t by $C(t)$, instantaneous utility by $U(C)$, and bliss by B, the objective function can be written

$$\text{Min} = \int_0^\infty \left\{ B - U(C(t)) \right\} dt \quad . \tag{1}$$

Let ρ stand for the output:capital ratio,[1] K for capital, $\overset{\circ}{K}$ for investment,[2] and Y for income. Then

$$Y = \rho K \tag{2}$$

$$Y = C + \overset{\circ}{K} \tag{3}$$

$$C = \rho K - \overset{\circ}{K}. \tag{4}$$

And expression (1) becomes

$$\text{Min} = \int_0^\infty \left\{ B - U(\rho K(t) - \overset{\circ}{K}(t)) \right\} dt \quad . \tag{5}$$

If we apply the calculus of variations to expression (5), the first order Euler-Lagrange equation becomes[3]

$$-\rho U_C = \overset{\circ}{U}_C \tag{6}$$

or

$$\rho = -\frac{\overset{\circ}{U}_C}{U_C} \quad . \tag{7}$$

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1 We shall assume throughout this essay that the production function is homogeneous of first degree, which means that ρ is a function of the labor:capital ratio alone.

2 Dots will in general indicate time rates of change.

3 Subscripts will in general indicate differentiation with respect to the variable indicated.

In view of the zero marginal productivity of labor associated with the optimal technique, the output:capital ratio ρ becomes equal to both the social and the physical marginal productivity of capital. But both may differ from the private marginal productivity of capital since a private computation of profit properly deducts any wage costs from the total return, despite the assumed redundancy of labor. The right hand side of (7) is the percentage rate at which the marginal utility of consumption falls over time, or the subjective rate of interest implied by society's utility function. Thus (7) expresses the Fisherian balance of opportunity and impatience in the determination of the optimal program of capital accumulation, although in the present instance the balance is one of social rather than private return with a social rather than a private subjective rate of interest.

Since the integral of (5) is a function only of K and $\overset{\circ}{K}$, we can integrate (6) to obtain a solution in terms of $\overset{\circ}{K}$:

$$\overset{\circ}{K} = \frac{B - U}{U_C} \tag{8}$$

Expression (8), the Ramsey-Keynes rule, says that the optimal rate of saving at any moment of time t is given by the ratio of difference between bliss and utility at t to the marginal utility of consumption at t.[1]

To give concreteness to (8) we shall adopt a specific form of the utility function, namely the constant elasticity function,

$$U(C) = -aC^{-v} \tag{9}$$

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[1] F. P. Ramsey, op. cit., p. 547.

where a and v are positive constants.[1] This function naturally suggests zero as the bliss-level, that is, $B = 0$. The marginal utility of consumption is given by

$$U_C = vaC^{-(v+1)} \tag{10}$$

and (8) becomes

$$\overset{o}{K} = \frac{0 - (-aC^{-v})}{vaC^{-(v+1)}} = \frac{C}{v} \quad . \tag{11}$$

Consumption plus savings are equal to total output, that is,

$$Y = C + \overset{o}{K} \tag{3}$$

1
 Cf. J. Tinbergen, "The Optimum Rate of Saving," Economic Journal, 66, 1956, 603; and "Optimum Savings and Utility Maximization Over Time," Econometrica, 28, 1960, 481. S. Chakravarty, "Optimal Savings With Finite Planning Horizon," International Economic Review, 3, 1962, 338. This utility function has simplicity to recommend it, but it also has the quality -- compelling to some and distressing to others -- of being the only utility function which implies that the subjective rate of interest depends only on the rate of growth of consumption and is independent of the level of consumption. A comprehensive discussion of the problems of defining a utility function in the context of infinite time can be found in S. Chakravarty, "The Existence of an Optimal Savings Program," Econometrica, 30, 1962, 178.

So, (11) is equivalent to

$$\frac{\mathring{K}}{Y} = \frac{1}{1+v} \, , \qquad (12)$$

In other words, the optimal saving rate $\mathring{K}/Y$ is constant over time and equal to the negative of the inverse elasticity of marginal utility with respect to consumption. Note that the optimal saving rate is independent of ρ.

For future reference we perhaps ought to specify society's subjective rate of interest (which henceforth we shall denote by r) implicit in the constant elasticity utility function. Division of the negative of the time rate of change of marginal utility,

$$-\mathring{U}_C = + (v+1)vaC^{-(v+2)} \mathring{C} \, ,$$

by the marginal utility of consumption (10) gives the subjective rate of interest,

$$r = -\frac{\mathring{U}_C}{U_C} = (v+1) \frac{\mathring{C}}{C} \, . \qquad (13)$$

It can be shown that (v+1) is the negative of the elasticity of marginal utility, and $\frac{\mathring{C}}{C}$ is the rate of growth of consumption. The subjective rate of interest is equal to their product. For any program of capital accumulation which maintains a constant savings rate s over time, the rate of growth of consumption is simply the product $s\rho$. Expression (13) becomes

$$r = (1+v)s\rho. \qquad (14)$$

Since v is fixed by tastes (those, let us say, of the planning commission, acting on behalf of "society") implementation of the Fisherian balance $r = \rho$ consists of choosing s equal to $(1+v)^{-1}$.

Saving and Technique Functionally Related

Now we can proceed to the heart of the present inquiry -- but not, unfortunately, without additional notation. Let w represent the exogenously fixed wage, and let ℓ denote the labor:capital ratio. Each value of ℓ is supposed to represent a different technique of production. The output:capital ratio ρ is a function of ℓ alone by virtue of the assumption of a first-degree homogeneous production function,[1] and we shall assume $\rho(\ell)$ is a strictly concave function, that is, one reflecting strictly diminishing marginal returns of output to labor. Let α stand for the proportion of profits (surpluses) that are saved, which will be assumed to be a decision under the control of the planning commission. Assuming that all wages are consumed, we have s as the following function of ℓ and α:

$$S(\ell,\alpha) = \frac{\alpha[\rho(\ell) - w\ell]}{\rho(\ell)} \quad . \tag{15}$$

In order to avoid mathematical complications, we shall limit our attention here to capital accumulation programs in which ℓ and α, and hence ρ and s, are fixed once and for all at time zero.

One extreme solution to the present problem is to proceed as before: to choose ℓ to maximize immediate output, that is, to maximize ρ — but subject now to the constraint imposed by labor's insistence on consumption,

$$\rho - w\ell \geq 0 \quad . \tag{16}$$

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[1] See footnote 2, page 6.

Maximization of the productivity of capital represents a direct application to the labor-surplus economy of the social marginal productivity (SMP) criterion of Alfred Kahn[1] and Hollis Chenery.[2] But it should be observed that the context in which the SMP criterion was advanced was not one in which the rate of saving was linked to the choice of technique. Maximization of ρ subject to the constraint embodied in expression (16) will -- if the constraint is binding -- lead to a zero rate of saving and hence a zero rate of growth of consumption. And precisely for this reason, the criterion of maximizing ρ is inapplicable under the present assumptions about the supply and behavior of labor. A decrease in the labor:capital ratio and the output:capital ratio in order to step up the savings ratio and the rate of growth of output and consumption seems clearly called for.

This suggests another extreme solution: to choose ℓ and α to maximize the rate of growth of output and consumption. This is indeed the criterion suggested by Walter Galenson and Harvey Leibenstein,[3] on the grounds that the maximal growth policy will eventually provide more consumption than any alternative program of capital accumulation. Maximization of the growth rate $s\rho$ clearly involves setting α equal to the boundary value of unity, and choosing ℓ to satisfy the first order condition

$$\frac{\partial(s\rho)}{\partial\ell} = \frac{\partial[\alpha(\rho-w\ell)]}{\partial\ell} = \alpha\,(\rho_\ell - w) = 0$$

or

$$\rho_\ell = w\,. \tag{17}$$

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[1] A. E. Kahn, "Investment Criteria in Development Programs," Quarterly Journal of Economics, 65, 1951, 38.

[2] H. Chenery, "The Application of Investment Criteria," Quarterly Journal of Economics, 67, 1953, 76.

[3] W. Galenson and H. Leibenstein, "Investment Criteria, Productivity and Economic Development," Quarterly Journal of Economics, 69, 1955, 343.

Maximization of the growth rate (which for $\alpha = 1$ is equal to the investible surplus per unit of capital) implies choosing ℓ to equate the marginal productivity of labor with the wage.[1] This corresponds, by the way, to choice of ℓ to maximize the return on capital as a state capitalist or private entrepreneur would measure it — output less wage costs. This solution suffers from the defect of sacrificing the present to the future regardless of how poor the present may become relative to the future in consequence, and regardless of how distant the future may be to which the present is sacrificed.

Maurice Dobb,[2] Otto Eckstein,[3] and Amartya Sen[4] have pointed out the extreme nature of these solutions, and each has sketched the outline of an alternative approach. Our own approach, choice of ℓ, α, s, and ρ in terms of utility maximization, is more in the spirit of Eckstein than of Dobb or Sen. As before, we suppose that instantaneous utility and consumption are related by the function

$$U(C) = -aC^{-v} \qquad a, v > 0 \tag{9}$$

Total utility $\mathcal{U}$ is given by

$$\mathcal{U} = \int_0^\infty U(C(t))dt = \int_0^\infty -a[C(t)]^{-v}dt \, . \tag{18}$$

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[1] The "marginal productivity of labor," unless qualified, means marginal productivity within the capitalistic sector; the marginal productivity of labor unassisted by capital is by assumption zero.

[2] M. Dobb, An Essay on Economic Growth and Planning, Monthly Review Press, 1960, Chapters 3 and 4.

[3] O. Eckstein, "Investment Criteria for Economic Development and the Theory of Intertemporal Welfare Economics," Quarterly Journal of Economics, 71, 1957, 55.

[4] A. K. Sen, "Some Notes on the Choice of Capital-Intensity in Development Planning," Quarterly Journal of Economics, 71, 1957, 560, and Choice of Techniques, Basil Blackwell, 1960.

With bliss taken as zero, Ramsey's objective of minimizing the integral of the difference between B and U is equivalent to maximization of $\mathcal{U}$.

Since we are confining our attention to once-and-for-all choice of t, a, s, and ρ, we can substitute for the equations

$$Y(t) = \rho K(t) \tag{2}$$

$$Y(t) = C(t) + \overset{\circ}{K}(t) \tag{3}$$

the equations

$$C(t) = (1-s)\,\rho K(t) \tag{19}$$

$$\overset{\circ}{K}(t) = s\rho K(t) \tag{20}$$

Integration of expression (20) gives

$$K(t) = K(0)\, e^{s\rho t} \tag{21}$$

where K(0) is the given initial capital stock. This also gives

$$C(t) = (1-s)\rho\, K(0)\, e^{s\rho t} \tag{22}$$

in place of (19).

If we substitute the right hand side of (22) for the left in expression (19) we have

$$\mathcal{U} = \int_0^\infty -a\,((1-s)\rho K(0)\, e^{s\rho t})^{-v}\, dt\ . \tag{23}$$

After integrating and substituting an equivalent expression for s from (15), equation (23) becomes

$$\mathcal{U} = \frac{-a[\rho-\alpha(\rho-w\ell)]^{-v} K(0)^{-v}}{v\alpha(\rho-w\ell)} \tag{24}$$

Maximization of $\mathcal{U}$ is equivalent to minimization of log $(-\mathcal{U})$ or to maximization of $-\log(-\mathcal{U})$, and this last is the easiest expression to work with. Now, $-\log(-\mathcal{U})$ is given by the equation

$$V = -\log(-\mathcal{U}) = \log a + v \log[\rho-\alpha(\rho-w\ell)] + v \log K(0) + \log v + \log \alpha + \log(\rho-w\ell) \quad . \tag{25}$$

Necessary conditions for maximization of (25) are given by

$$\frac{\partial V}{\partial \alpha} \begin{Bmatrix} \geq \\ = \end{Bmatrix} 0 \text{ as } \begin{Bmatrix} \alpha = 1 \\ \alpha < 1 \end{Bmatrix} \tag{26}$$

and

$$\frac{\partial V}{\partial \ell} = 0 \quad . \tag{27}$$ [1]

[1] The boundary value $\alpha = 0$, which would correspond to $\frac{\partial V}{\partial \alpha} \leq 0$, and the SMP choice of ℓ (ℓ such that $\rho = w\ell$), which would correspond to $\frac{\partial V}{\partial \ell} \geq 0$, can be eliminated simply by virtue of the fact that either of these choices would lead to the dominated utility value $\mathcal{U} = -\infty$. The possibility of $\frac{\partial V}{\partial \ell} <$ for the Galenson-Leibenstein choice of ℓ (ℓ such that $\rho_\ell = w$), can be ruled out by appealing to continuity: Since $\frac{\partial V}{\partial \ell}$ is positive for all values of less than the Galenson-Leibenstein value, it cannot be negative at the Galenson-Leibenstein value.

From (25) we have

$$\frac{\partial V}{\partial \alpha} = \frac{v(\rho - w\ell)}{\rho - \alpha(\rho - w\ell)} + \frac{1}{\alpha}$$

Thus (26) becomes

$$\rho \begin{Bmatrix} \geq \\ = \end{Bmatrix} (1+v)\ \alpha\ (\rho - w\ell) \qquad \text{as} \begin{Bmatrix} \alpha = 1 \\ \alpha < 1 \end{Bmatrix} \tag{28}$$

Now combining (14) and (15) gives

$$r = (1+v)\alpha(\rho - w\ell) \quad , \tag{29}$$

so that (28) becomes

$$\rho \begin{Bmatrix} \geq \\ = \end{Bmatrix} r \quad \text{as} \quad \begin{Bmatrix} \alpha = 1 \\ \alpha < 1 \end{Bmatrix} \tag{30}$$

The derivative $\frac{\partial V}{\partial \ell}$ is given by

$$\frac{\partial V}{\partial \ell} = \frac{v[\rho_\ell - \alpha(\rho_\ell - w)]}{\rho - \alpha(\rho - w\ell)} + \frac{\rho_\ell - w}{\rho - w\ell} \tag{31}$$

Thus, equation (27) becomes

$$\rho - \rho_\ell \ell = (1+v)\alpha(\rho - w\ell)\frac{\alpha(w-\rho_\ell)+\rho_\ell}{\alpha w} = r\ \frac{\alpha(w-\rho_\ell) + \rho_\ell}{\alpha w} \quad . \tag{32}$$

Since the left-hand side of (32) is smaller than ρ unless $\rho_\ell = 0$, and the right-hand side is greater than or equal to r, the equality in (28) can hold only in the event $\rho_\ell = 0$, in which case (32) as well as (30) reduce to the Euler-Lagrange equation, (7), that characterizes the optimal growth path under the assumption that savings and technique are divorced. This should not be surprising, for if zero marginal productivity of labor is consistent with the optimal solution in the present problem, the constraint on savings imposed by the consumption of wage income is in fact not binding, and the present problem reduces to the previous one, in which technique and savings can be independently optimized .

If the solution of (32) requires $\rho_\ell > 0$, the constraint arising from the consumption of wage income limits the choice of savings rate and the choice of technique, and strict inequality must hold in (30). This is to say that α must equal unity; in other words, optimal growth requires reinvestment of all surpluses remaining after payment of the instiutionally fixed wage bill, $w\ell K$. In this case, the optimal technique is given by the value of ℓ for which equation (33) holds:

$$\rho - \rho_\ell \ell = r \quad . \tag{33}$$

This equation is (31) with α replaced by unity.

Equation (33) reflects a balance between "opportunity" and "impatience" when consumption of wage income is an effective constraint on choice of technique and savings rate. Marginal impatience is reflected in the value of the subjective rate of interest r. Opportunity is here represented by the physical marginal productivity of capital,

$$\left(\frac{\partial Y}{\partial K}\right)_{\text{employment = const.}} = \rho - \rho_\ell \ell ,$$

the extra output from an extra unit of capital with employment unchanged. The physical productivity should be distinguished from the marginal social productivity of capital,

$$\left(\frac{\delta Y}{\delta K}\right)_{\ell = \text{const.}} = \rho \ ,$$

which measures the extra output from an extra unit of capital when employment is increased to maintain a constant labor:capital ratio, which is the optimal employment strategy from the point of view of social utility maximization. The physical productivity of capital is equal to the return on capital measured by subtracting from the social productivity a labor cost computed by replacing the wage w with a lower shadow wage equal to the marginal productivity of labor, ρ_ℓ.[1] For the optimal technique and savings rate the following inequality holds:

$$\rho > \rho - \rho_\ell \ell = r > \rho - w\ell = s\rho \qquad (34)$$

Note that as v approaches zero, the $\mathcal{U}$-maximizing choice of ℓ approaches the Galenson-Leibenstein growth-maximizing choice, and the physical productivity of capital and subjective rate of interest approach the private rate of return to capital: $v \to 0$ implies $r \to (\rho - w\ell)$; hence from (33) $\rho_\ell \to w$, which implies maximization of the rate of growth. A similar argument establishes that as $v \to \infty$, the optimal choice of ℓ approaches the SMP choice of ℓ to equate ρ with $w\ell$ (unless a smaller value of ℓ drives ρ_ℓ to zero).

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1 The shadow wage can be expressed in terms of w by substituting in (33) an equivalent expression for ℓ from (15). This substitution gives

$$\rho_\ell = \frac{(\rho - r)w}{(1-s)\rho} < w$$

If we momentarily change the ground rules, and assume that α is a parameter fixed exogenously rather than a choice variable, equality between the subjective rate of interest and the physical productivity of capital no longer characterizes the socially optimal choice of technique. In this case, which corresponds to a mixed economy in which the government controls employment but not savings (the value of α being determined, for example, by the behavior of private capitalists, just as the consumption of wage income is determined by the behavior of workers), equation (32) alone characterizes the optimum, and the optimal physical productivity of capital exceeds the subjective rate of interest. The ratio of the physical productivity of capital to the subjective rate of interest,

$$\frac{\alpha w + (1-\alpha)\rho_\ell}{\alpha w},$$

varies inversely with the propensity to save of capitalists.

A numerical example might be useful in assessing the difference between optimization in terms of utility maximization and optimization in terms of the alternative criteria of choice to which reference has been made. Suppose production is governed by the Cobb-Douglas function (with L representing employment and the other variables defined as before)

$$Y = L^{1/2} K^{1/2} = \ell^{1/2} K$$

so that

$$\rho = \ell^{1/2}$$

Further, let $v = 2$, and let $w = 2$. Then, Table One gives the values of the several variables associated with utility maximization and, for contrast, with maximization of immediate output and with maximization of the rate of growth.

Table One

Parameter values resulting from application of alternative criteria with $\rho = \ell^{1/2}$, $v = 2$, $w = 2$.

	Criterion		
	Max ρ	Max $\mathcal{U}$	Max $s\rho$
α = proportion of profits saved	-*	1.0	1.0
ℓ = labor:capital ratio	0.25	0.173	0.0625
ρ = output:capital ratio = social productivity of capital	0.5	0.416	0.25
s = rate of saving	0.0	0.166	0.5
$s\rho$ = $\rho - w\ell$ = rate of growth = private rate of return on capital**	0.0	0.0695	0.125
ρ_ℓ = marginal productivity of labor = shadow wage	1.0	1.2	2.0
$\rho - \rho_\ell \ell$ = physical productivity of capital	0.25	0.208	0.125
r = subjective rate of interest	0.0	0.208	0.375

* The value of α is irrelevant since $\rho - w\ell = 0$

** Since either $\alpha = 1$ or $\rho - w\ell = 0$, it follows that $s\rho = \rho - w\ell$.

It should be observed in Table One that the physical productivity of capital and the subjective rate of interest are equal only for the optimal growth path "max $\mathcal{U}$." For growth rates less than optimal, of which the "max ρ" path is an extreme example, the physical productivity of capital exceeds the rate of interest; for growth rates greater than optimal, of which the "max $s\rho$" path is the limiting case, the opposite is true.

Figure one illustrates some of the magnitudes of Table One graphically. The next three figures indicate the time profiles of output, consumption, and employment resulting from the three criteria. Initial capital stock K(0) is assumed in all cases to be equal to 100.

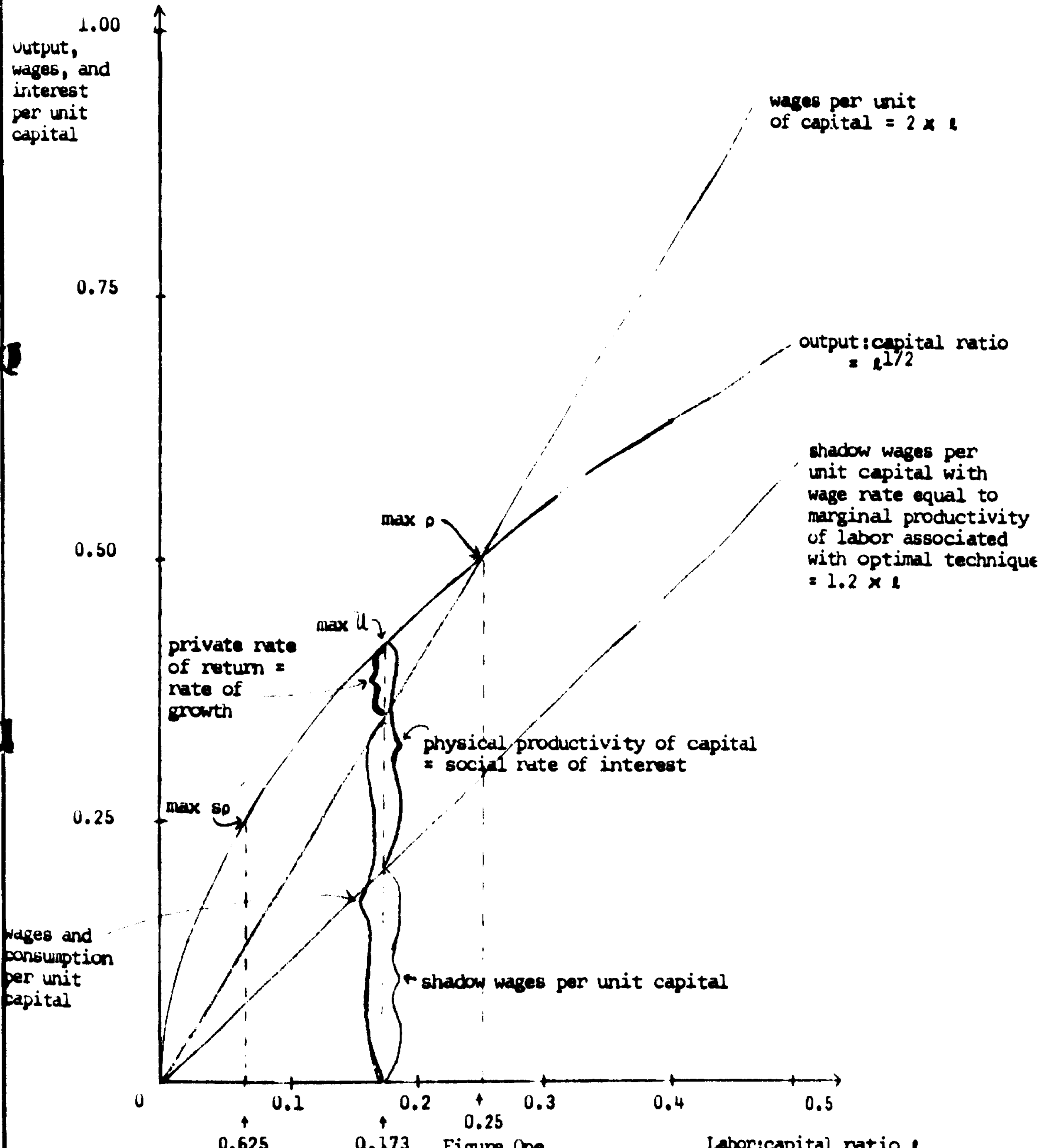


Figure One

Output, wages, and interest as function of technique of production.

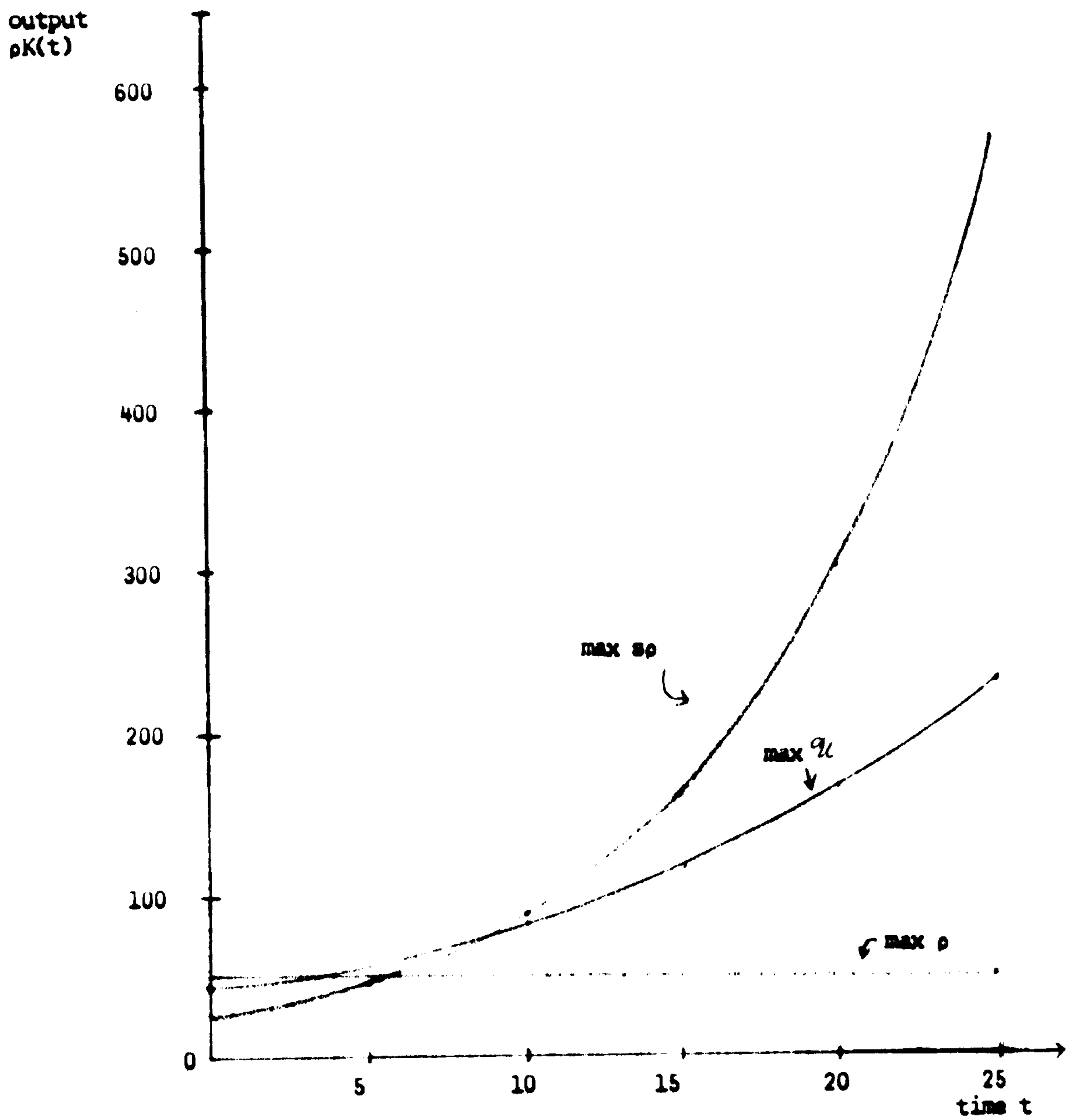
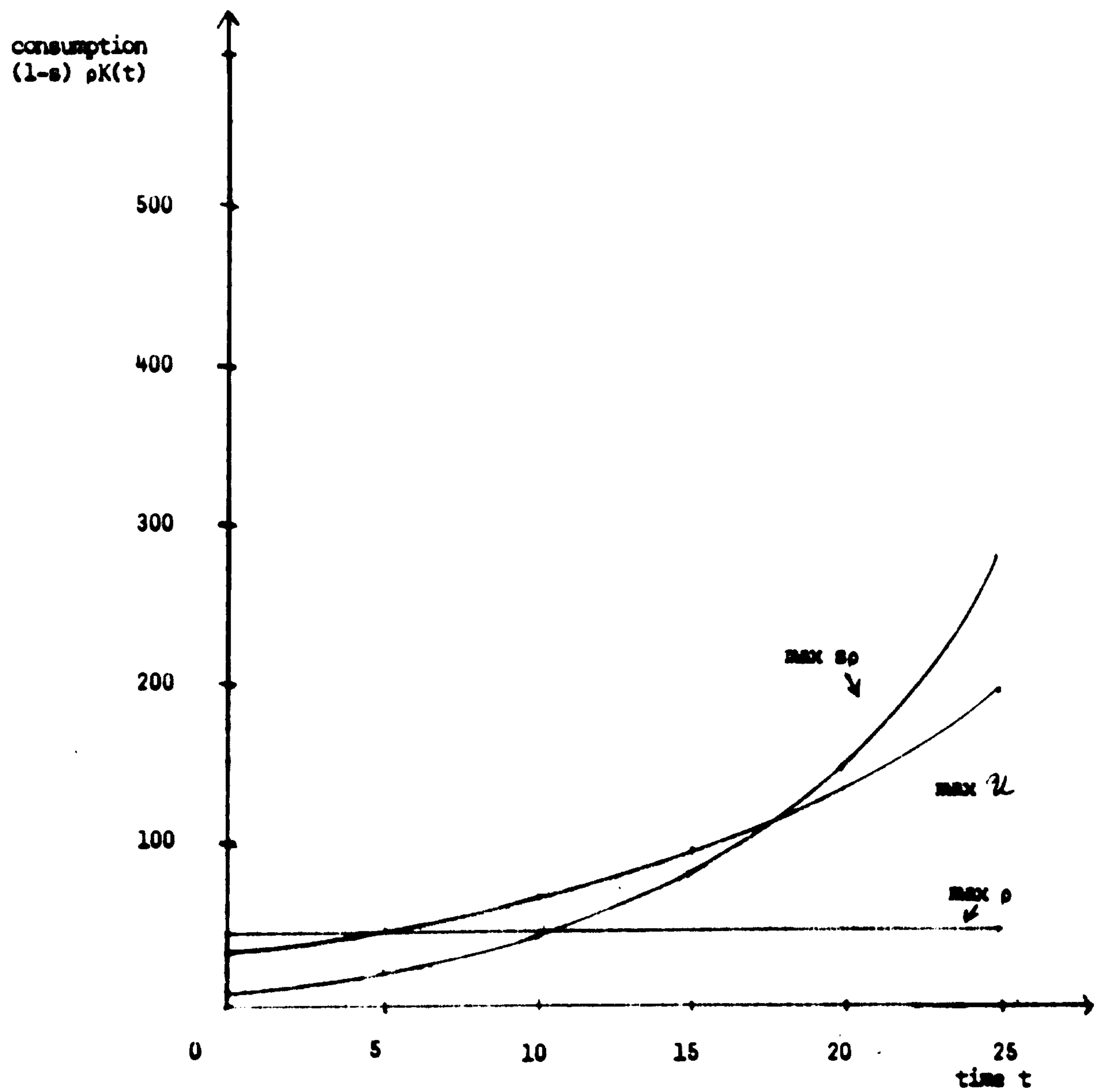


Figure Two

Time profile of output resulting from utility maximization and from alternative criteria



<u>Figure Three</u>

Time profile of consumption resulting from utility maximization and from alternative criteria

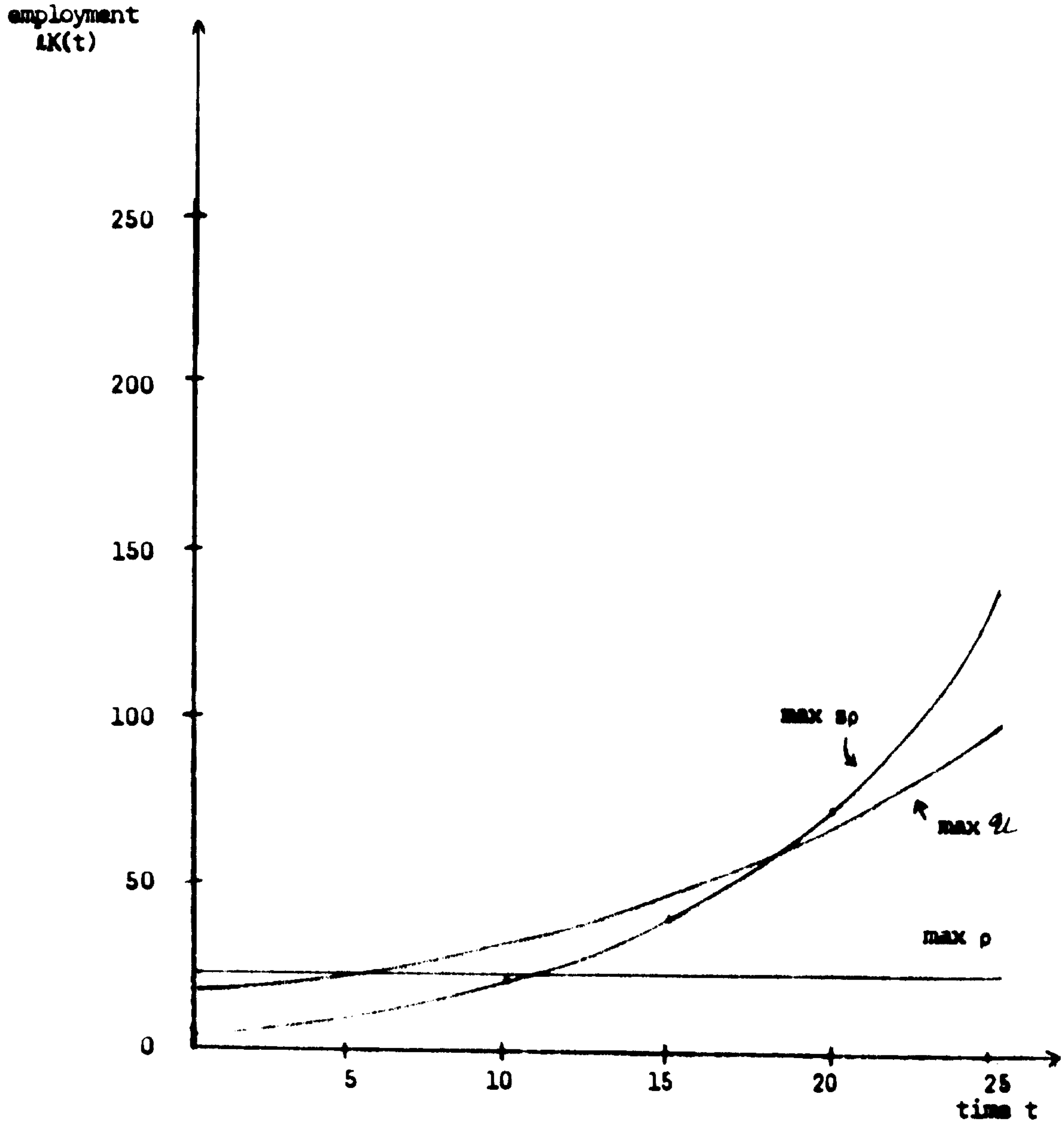


Figure Four

Time profile of employment resulting from utility maximization and from alternative criteria

The Value of Capital

In the economy of our model, output and consumption are governed by the simple relationships

$$Y(t) = \rho K(t) = \rho K(0)\, e^{s\rho t}$$

$$C(t) = (1-s)\rho K(t) = (1-s)\rho K(0) e^{s\rho t}$$

But suppose we relax this assumption slightly to allow the planning commission to be presented with the possibility of an alternative use of one unit of capital at time t_o. The time pattern of consumption provided by the new opportunity, let us suppose, is given by the function $\Delta(t)$, $t_o \le t < \infty$. (This function is assumed to reflect reinvestment of surpluses over wage costs.) The choice facing the planners is whether or not to divert one unit of investment to the new option when the opportunity arises.

How might planners make this decision? The first step is to compute the marginal utility afforded by the new opportunity. Denoting this marginal utility by $\mathcal{U}_\Delta$, we can write

$$\mathcal{U}_\Delta = \int_{t_o}^{\infty} U_C \Delta(t)\, dt = \int_{t_o}^{\infty} U_{C(0)} \Delta(t) e^{-rt} dt$$

$$= U_{C(0)} e^{-rt_o} \int_{t_o}^{\infty} \Delta(t) e^{-r(t-t_o)} dt$$

where $U_{C(0)}$ equals the marginal utility of consumption at time $t=0$ and (by virtue of the constancy of the elasticity of utility)

$$U_C = U_{C(0)} e^{-rt} .$$

The second step is to compare $\mathcal{U}_\Delta$ with the marginal utility of investment at time t in the optimal technique as determined by v, w, $\rho(\ell)$ and $s(\ell,\alpha)$. This marginal utility we denote $\mathcal{U}_{K(t_o)}$. We have

$$K(t_o) = \int_{t_o}^{\infty} U_C C_{K=K(t_o)} dt = U_{C(0)} e^{-rt_o} (1-s)\rho \int_{t_o}^{\infty} e^{(s\rho-r)(t-t_o)} dt = U_{C(0)} e^{-rt_o} \frac{(1-s)\rho}{r-s\rho}$$

For the $\mathcal{U}$-maximizing choices of s and ℓ, substitution from (15) and (33) gives the equality

$$\mathcal{U}_{K(t_o)} = U_{C(0)} e^{-rt} \frac{w}{w-\rho_\ell} \tag{36}$$

Now the new opportunity should be exploited only if $\mathcal{U}_\Delta > \mathcal{U}_{K(t_o)}$, or in other words, only if

$$U_{C(0)} e^{-rt_o} \int_{t_o}^{\infty} \Delta(t) e^{-r(t-t_o)} dt > U_{C(0)} e^{-rt_o} \frac{w}{w-\rho_\ell} \tag{37}$$

This criterion can be made a little more familiar by normalizing by means of division of (37) by $U_{C(t_o)} = U_{C(0)} e^{-rt_o}$, that is, by dividing both sides of (37) by the marginal utility of consumption at time t_o. Then

$$\frac{\mathcal{U}_\Delta}{U_{C(t_o)}} = \int_{t_o}^{\infty} \Delta(t) e^{-r(t-t_o)} dt \tag{38}$$

$$\frac{\mathcal{U}_{K(t_o)}}{U_{C(t_o)}} = \frac{w}{w-\rho_\ell} \tag{39}$$

And the criterion of superiority of the new use of capital over the "optimal" technique, expression (37) becomes

$$\int_{t_0}^{\infty} \Delta(t)e^{-r(t-t_0)}dt > \frac{w}{w-\rho_\ell} \qquad (40)$$

The integral $\int_{t_0}^{\infty} \Delta(t)e^{r(t-t_0)}dt$ is customarily called the present value at time t_0 of the consumption stream $\Delta(t)$ evaluated at the discount rate r. Similarly

$$\frac{K(t_0)}{U_{C(t_0)}} = \frac{w}{w-\rho_\ell} \qquad (41)$$

is the marginal present value of investment in the "optimal" technique. Thus (40) says that the present value of the new opportunity must exceed the marginal present value of investment in the "optimal" technique. This may be a bit surprising, for the physical trade-off rate between consumption and investment determined by the equation

$$Y(t) = C(t)+\overset{\circ}{K}(t) \qquad (3)$$

is unity, and we might therefore have expected that the new investment option would be attractive provided its present value, $\int_{t_0}^{\infty} \Delta(t)e^{-r(t-t_0)}dt$, exceeded unity. But in the present model, limitations on the choice of s mean that the marginal rate of substitution, as reflected in the marginal present value of investment, is in excess of the technological transformation rate, and it is with the first rather than the second that the present value of alternative $\int_{t_0}^{\infty} \Delta(t)e^{-r(t-t_0)}dt$ must be compared.

The marginal present value of investment $w/(w-\rho_\ell)$ is thus the shadow price of investment. Since in the present model average and marginal values coincide, $w/(w-\rho_\ell)$ is also the shadow price of capital. This shadow price falls to unity only in the limiting case of $\rho_\ell = 0$ and $\rho = r$, when the production function $\rho(\ell)$, the elasticity of utility $-v$, and the wage rate w combine to make it possible to divorce the savings question from the technique question. At the other extreme, when v goes to zero and $\mathcal{U}$- maximization dictates choosing ℓ to provide a rate of growth of output and consumption that approaches the maximal feasible rate of growth, ρ_ℓ goes to w and the shadow price of capital approaches infinity.

Measurement of the effectiveness of potential investments thus requires a more elaborate evaluation than would be necessary were it not for labor's effective insistence on consumption. Because the choice of the rate of saving cannot be divorced from the choice of technique, investment planning requires not only specification of a discount rate but also specification of a shadow price of capital. The present value of consumption stream resulting from each investment opportunity (including whatever consumption is afforded by reinvestment) must be computed at the social rate of discount and this present value compared with the capital cost computed with a shadow price equal to the economy's marginal present value of capital. Only in the event that $\rho = r$ for the optimal technique and the conflict between savings and growth, on the one hand, and immediate output, consumption, and employment, on the other, disappears does this evaluation procedure reduce to the more familiar procedure of comparing discounted present value with the nominal capital cost.

Decentralization of Choice of Technique

It is evident that laissez-faire cannot be relied upon to produce the optimal technique under the assumptions of the model. Decentralized entrepreneurs left to their own devices would maximize the private rate of return to capital, $\rho - w\ell$, as would profit-maximizing state capitalists.

Decentralized "market socialists" of a Lange-Lerner type could be guided to the $\mathcal{U}$-maximizing choice of technique by an order from the planning commission to choose the technique of production to maximize the physical rate of return, $\rho - \rho_\ell \ell$, computed with a shadow wage equal to the marginal productivity of labor associated with the optimal technqiue. This instruction would have to be supplemented by an order to reinvest all surpluses remaining after actual wage costs are paid.

Replacing w by ρ_ℓ in choice of technique calculations amounts to an "as if" subsidy of $w - \rho_\ell$ per unit of labor. Choice of the optimal technique could be achieved through payment of an actual subsidy of $w - \rho_\ell$ to private entrepreneurs or state capitalists; but the taxes levied to pay the subsidy must not fall on the workers, for this would violate the rules of the game, which requires consumption of all wage income.[1]

The difficulty with decentralization of decision-making on the basis of a shadow wage is a familiar one: the optimal technique must be known to the planning commission in order to determine the appropriate shadow wage. Hence, there might seem to be little advantage in decentralization. However, the optimal technique and shadow wage could be determined simultaneously by a decentralized tâtonnement

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[1] The indirect control exercised through the subsidy of wages would have to be supplemented by direct control to ensure reinvestment of all profits remaining after payment of wages and taxes. But private capitalists would presumably tire very quickly of always having their cake and never eating it.

procedure. If ℓ is iteratively adjusted according to the formula

$$\ell^{n+1}-\ell^{n} = -\theta(\rho^{n}-\rho_{\ell}^{n}\ell^{n}-r^{n}) \qquad n = 1, 2, 3, \ldots \qquad (42)$$

where ρ^n, ρ_ℓ^n and r^n are values associated with $\ell = \ell^n$ and θ is a positive constant then convergence of the sequence $\{\ell^n\}$ to an arbitrarily small neighborhood of the optimal labor:capital ratio can be guaranteed by suitable choice of θ regardless of the initial choice of ℓ.[1] Equation (42) says in effect that the labor:capital ratio should be decreased (in order to increase the rates of saving and growth) so long as the physical productivity of capital exceeds the social rate of discount, and vice-versa. The social rate of interest would be recomputed from (14) by the planning commission between iterations and transmitted to the decentralized managers, who after computing the values of ρ and ρ_ℓ would calculate the new value of ℓ from equation (42) and transmit the associated values of ρ and s to the planning commission. This would in turn suggest a new value of r, which would form the basis for the next iteration.

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[1] This proposition presupposes that the optimal technique implies $\alpha = \ell$. (The more general case can be covered by suitably amending the algorithm embodied in (42). The proof of the convergence of the sequence defined by (42) to an arbitrary small neighborhood of the $\mathcal{U}$-maximizing value of ℓ requires nothing more than modification of the proof of convergence of a continuous gradient process to allow for discrete changes in the values of variables. See K. J. Arrow, L. Hurwicz and H. Uzawa, Studies in Linear and Non-Linear Programming, Stanford University Press, 1958, Chapter 10.

Conclusions

The basic assumptions of the model explored in this essay are (1) the availability of unlimited supplies of labor in the perpetuity at an exogenously determined wage rate and (2) the consumption of all wage-income. A third assumption is that labor neither involves disutility nor is productive without the assistance of capital. Without assumption (2), the choice of technique is a relatively simple affair: the goal is clearly to choose the labor:capital ratio ℓ to maximize the output:capital ratio ρ. In this case the choice of a rate of saving s (which together with ρ determines the rate of growth of output, consumption, and employment) is a separate question. But insistence on the part of labor on consumption of its entire income makes it impossible to divorce the choice of technique from an upper bound on s; savings now can come only from profits. The greater the value of ℓ and ρ (beyond the point where the marginal productivity of labor ρ_ℓ falls to the level of the wage rate w), the lower is the upper limit on s. Others[1] have explored the conflict between immediate output and the rate of growth that the dependence of s on ℓ poses, and it has been pointed out that in general the optimal technique can be expected to reflect a compromise between the maximal feasible immediate output and the maximal feasible rate of growth. The present analysis, couched in terms of maximization of an explicit utility function (chosen for convenience to reflect a constant elasticity with respect to consumption) confirms the wisdom of compromise, but our chief interest has been not in the compromise itself but rather in its implications with respect to wages and interest.

The principal conclusion was stated at the outset of this essay, but certainly bears repeating: Neither the social rate of return (or social marginal productivity)

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[1] M. Dobb, op. cit.; O. Eckstein, op. cit.; and A. K. Sen, op. cit.;

of capital ρ nor the private rate of return $\rho - w\ell$ is equal to the subjective rate of interest r that reflects the marginal premium on present over future consumption that is implicit in the economy's utility function -- even for the optimal technique and saving rate. The Fisherian balance of opportunity and impatience characterizing utility maximization is implemented instead by the following equality between the physical marginal productivity of capital and the subjective rate of interest:

$$\rho - \rho_{\ell}\ell = r \qquad (33)$$

The physical marginal productivity of capital on the left hand side of (33) is equivalent to the yield on capital measured by subtracting labor costs evaluated on the basis of a shadow wage (equal to the marginal productivity of labor associated with the optimal technique) from the output:capital ratio. Furthermore, the marginal productivity of labor optimally lies between zero and the actual wage, so that

$$\rho \geq \rho - \rho_{\ell}\ell = r > \rho - w\ell \qquad (34)$$

The private rate of return $\rho - w\ell$ is equal to the rate of growth of output, consumption, and employment, $s\rho$, provided all surpluses remaining after payment of wages are reinvested,[1] so that (34) can be interpreted as setting upper and lower bounds for the rate of interest as, respectively, the output:capital ratio and the rate of growth of the economy. The rate of interest will actually attain the upper bound only in the event the technology is such that it permits the best of both worlds simultaneously — the maximum output:capital ratio (which implies $\rho_{\ell} = 0$) and independent optimization with respect to the rate of saving.

1 Reinvestment of all surpluses turns out to be a condition of optimality except in the limiting case $\rho = r$, in which event the conflict between immediate output and the rate of growth disappears.

The rate of interest appropriate for discounting the consumption stream generated by any new investment opportunities that may be afforded from time to time is r, for discounting at r is equivalent to weighting consumption at each moment of time by its marginal utility. But the decision whether or not to undertake any such investment cannot be made by comparing the present value of its consumption stream at r with its capital cost. The inability of the economy to optimize independently with respect to the rate of saving means that the marginal rate of substitution of consumption for investment, in other words, the marginal present value of investment at the social rate of discount, exceeds the physical rate of transformation of unity at "equilibrium". The present value afforded by any investment opportunity must therefore be compared with its capital cost evaluated at a shadow price equal to the marginal present value of investment in the economy. This marginal present value falls to unity only in the event that $\rho = r$ and the conflict between immediate output and employment, on the one hand, and savings and growth, on the other disappears.[1]

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[1] The point is a general one. When institutional constraints of any kind prevent optimization with respect to the rate of saving, the social, private and physical productivities of capital will in general differ, and the price, or "opportunity cost" of capital will differ from the purely physical marginal rate of transformation between consumption and investment goods. The question of interest rates and capital valuation for purposes of public investment is explored from a basis that reflects the conditions of mature mixed-enterprise economies rather than the destructive labor-surplus feature of underdeveloped economies in two articles: S. A. Marglin, "The Social Rate of Discount and the Optimal Rate of Investment," Quarterly Journal of Economics, 77, 1963, 95, and "The Opportunity Costs of Public Investment," Quarterly Journal of Economics, 77, 1963, 274.

Because of the difference between the private rate of return and the social rate of discount, laissez-faire could not be expected to lead to an optimal choice of technology. A subsidy on labor costs to private entrepreneurs or state capitalists, or an "as if" subsidy to market socialists, would, however, make private and shadow returns coincide. In principle, the size of the subsidy $w-\rho_{\ell}$, with ρ_{ℓ} the marginal productivity of labor associated with the optimal technique, can be determined along with the optimal technique by a decentralized tatonnement as well as by centralized planning.

The model on which the conclusions of this essay are based is an extremely simple one. It ignores the existence of a multiplicity of sectors, technologies, and outputs in the economy. It ignores foreign trade. It assumes unlimited supplies of labor not simply for the present but in perpetuity. It assumes absolute rigidity with respect to real wage rates and consumption by workers. Moreover, the choice of technique and savings rate are posited as once and for all decisions. Finally, the utility function chosen -- besides being extremely simple with respect to total consumption -- does not take distribution of consumption into account at all, and distribution is surely an important aspect of the conflict of immediate output and employment against savings and growth. Nevertheless, the propositions we have sought to establish are qualitative rather than quantitative in nature, and for this purpose a simple model suffices as well as a complex one. The precise form of the conclusions will certainly be affected by added doses of realism, but not their nature.

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