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United Nations
Centre for Industrial Development

Original: English

Inter-Regional Symposium on Industrial Project Evaluation

CID/IPE/B.28
Discussion Paper

Prague, Czechoslovakia
11 - 29 October, 1965

07582

PROJECT EVALUATION IN THE PRESENCE OF ECONOMIES OF SCALE AND INDIVISIBILITIES

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65-41904

A. Economic Equilibrium versus Non-convexity

Almost all existing or proposed techniques of project evaluation[1/] are based directly or indirectly on the notion of economic equilibrium, i.e., on the notion of a feedback mechanism that adjusts all prices and quantities in an economic system in such a way that on the one hand, demands and supplies of all goods and factors and on the other, revenues and costs of all economic activities - production, transport, storage, training, etc. - are equilibrated.

In its original version, this notion of economic equilibrium was proposed as an explanation of the behaviour of actual markets under free enterprise; later, as the shortcomings of the market mechanism - monopoly elements, limited effective demand, unsatisfactory distribution of income and wealth, frustrated growth - became widely recognized, equilibrium was still held up as an ideal which could be approximated in practice to a "reasonable" or "workable" extent. Lately, with the advent of mathematical programming techniques, it has become possible to isolate the notion of economic equilibrium completely from the behaviour of actual markets, and to replace the latter by electronic computer solutions to planning models with varying degrees of centralization or decentralization. In fact, the notion of economic equilibrium can be extended by means of computer solutions to models representing many economic situations that even ideally competitive markets would be unable to realize in practice: for example, multi-period

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1/ On investment criteria in economic planning, see for example Chenery (1953:AIC), Bohr (1954), Galenson and Leibenstein (1955), Eckstein (1957), UN-Manual of Economic Development Projects (1958), UN-Div.Ind.Dev. (1963:EPP).

(dynamic, growth) models with terminal conditions imposed;[2/] resource allocation models with limits set on the ability of the government to execute certain policies such as wage subsidization, and others.

It has been known for a long time, however, that the notion of economic equilibrium has a limitation that cannot be overcome by minor modifications. This was recognized at first as the problem of fixed costs, i.e. of diminishing average costs as the scale of production increases. It has proven to be generally impossible to reconcile the requirements of efficient resource allocation (marginal-cost pricing) with the need for covering the fixed costs incurred by the firm out of revenues obtained from product sales, unless two conditions are fulfilled:

(i) Average costs, while falling initially as a result of the distribution of fixed costs over a larger number of units, eventually level out (become horizontal) or even turn up, due to elements of increasing variable cost that offset (or outweigh) diminishing unit fixed costs.

(ii) The critical scale at which the average costs of the firm level out is much smaller than total production within the industry

Under the above two conditions industry supply can be taken as the sum of the critical scales of successive firms; insofar as these critical scales are much smaller than total production within the industry, supply can be approximated by a continuous function even though, in actual fact, this supply satisfies equilibrium conditions only at selected lattice points representing the exact sums of critical scales.

2/ Note that the so-called "dynamic invisible hand" theorem (see Dorfman, Samuelson and Solow, 1958, p. 319) that extends the principle of social efficiency of perfectly competitive markets from a static to a dynamic context guarantees only that such a system, once locked on an efficient growth path, will stay on it; but it cannot direct the system toward a growth path that satisfies exogenously determined terminal social objectives.

It can thus be seen that even under highly idealized conditions, the so-called equilibrium solution is only an approximation to what is now recognized as the optimal solution to a mathematical problem known as integer programming.[3] Recent theoretical advances in the latter field, moreover, throw doubt upon the validity of any such approximation, since they indicate that the solution to an integer programming problem that is obtained by rounding (to the nearest integer) the optimal solution to a continuous approximation will generally not be an optimal solution to the integer problem.[4]

The situation, of course, becomes even less satisfactory to the extent that the two basic assumptions are not satisfied. If assumption (i) is satisfied (levelling out of the average cost curve) but assumption (ii) is progressively weakened (larger critical firm size in relation to industry production), then the process of rounding to the nearest integer will imply larger and larger percentage changes with respect to the continuous solution, and a greater possibility that the rounded solution will be strongly sub-optimal. Eventually, as industry production falls below the critical size of a single firm, there will be a frank contradiction between the customary marginal-type efficiency conditions and the recovery of fixed costs through sales revenues. The same result is obtained whenever assumption (i) is dropped.

3/ For a survey see for example Dantzig (1963), chap. 26; see also Gomory (1963, 1965).

4/ Gomory (1965).

The presence of fixed costs is a case of mathematical "non-convexity" leading to economies of scale.[5] Such economies of scale can also occur in the absence of actual fixed costs, depending on the shape of the production function.[6] Other cases of non-convexity of interest to economic planning are :

- indivisibilities: the necessity of planning in multiples of standardized production units; zero-one decisions on transport investments, hydroelectric projects, etc.;[7]
- pre-emption of land area: the fact that a given plot of ground (e.g. in a densely occupied zone) has to be assigned in a zero-one fashion to individual uses;[8]
- either/or type constraints on feasible policy alternatives, etc.[9]

It has come to be recognized that a decentralized decision-making system based on linear decentralizing instruments (master prices, administratively determined planning prices, incentive systems with linear structure) is inherently unable to guarantee attainment of an optimal equilibrium position unless all sources of non-convexity - such as fixed costs and others - are absent. Therefore no project evaluation

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5/ A point set [illegible] is convex if the following holds: if [illegible] and λ_i [illegible] and [illegible] $\lambda_i = 1$ then [illegible] x_i [illegible], where i=1,...,n. Applied to an available technology consisting of a collection of projects this concept of convexity means that any weighted average of technically feasible individual projects will also be technically feasible. Note that where economies of scale are present convexity breaks down. For example, if the actual capital input requirements of a process comprise a fixed input plus an input proportional to scale, then two half-sized projects using this process will actually use more capital than one full-size project; in other words, the average of two half-sized projects (with equal weights) will underestimate capital requirements and will thus describe an infeasible technology.

criteria that are based on the notion of economic equilibrium and involve correspondingly any linear version of pricing or other decentralizing/control systems, whether these be market prices, corrected opportunity costs, electronically computed shadow prices based on mathematical programming models, or administratively fixed prices in a planned economy 10/ can be relied upon with confidence in the presence of non-convexities. As the case may be, they can turn out to yield acceptable results, but they can equally well result in gross misallocations.

Two illustrations will indicate the kinds of outcomes that are possible when linear decentralizing instruments are used in the presence of nonconvexities. Chenery in <u>The Interdependence of Investment Decisions</u> (1959) constructs a detailed numerical example of steel production and iron-ore mining with strong economies of scale in a developing country. The analysis reveals that either one of these two activities is profitable when the other activity is present, but is unprofitable in its absence; thus a decentralized decision system based on profit (or social marginal product) misses an attractive <u>joint</u> investment opportunity.

6/ Economies of scale often are expressed by an input function of the form:

$$(y/\bar{y}) = (x/\bar{x})^f,$$

where y and $\bar{y}$ are inputs corresponding to scales x and $\bar{x}$; the barred quantities are constants; and f is a constant exponent in the range $0 < f \leq 1$

7/ See Victoris (1964).

8/ Koopmans and Beckman (1957).

9/ See Dantzig (1960).

10/ For a discussion of different kinds of profitability indexes used as decentralising instruments in a centrally planned economy, see Kornai and Liptak (1962).

When neither of these activities is yet established the decentralized decision maker looking at an activity in isolation will decide that it is unprofitable; thus neither of the two activities can historically precede the other and the profitable complex of the two activities will never be attained.[11] Koopmans and Beckman in Assignment Problems and the Location of Economic Activities (1957) construct an example which shows nonconvexities involved in the assignment of productive activities to discrete locations that cannot be shared between activities. For example, in an urban area a given block or plot of land can not be used for both a large shopping center and an industrial plant. In many locational problems no such assignments are required; for example, if productive locations have to be chosen for industries that can locate at several regional centers that are at large distances from each other, the land requirements at these centers are usually very small in comparison with the available industrial sites and thus several activities may easily locate at the same center. The latter kind of locational problems are generally convex (unless economies of scale occur independently in the production or transport activities) and a stable price system exists that can be utilized for the definition of project evaluation criteria in the usual way. In the former locational assignment problem, however, the present location of any activity will affect the costs of all other activities in such a way that with any locational pattern incentives will exist for some producers to change their locations, and the possibility of a stable equilibrium price system is negated.

When significant non-convexities are known to be present -- important industrial processes whose optimal scales of operation are not attained at the level of demand of a small country, important decisions concerning investments in transport arteries, etc. - the only reliable

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11/ There have been numerous qualitative discussions of the interrelations between industries in the course of economic development due to economies of scale and externalities. Economies of scale create technical interrelations such as discussed by Chenery; they also lead to complementarity between industries producing consumer goods. External economies arise in education, labor training and activities aimed at securing technical progress; in social-overhead investments (Cont'd) approach to the evaluation of individual projects is an overall analysis of all alternative projects within the framework of a mathematical programming model in which non-convexities are explicitly accounted for. Finding the optimal solutions to such models is an analytical problem which is not yet satisfactorily resolved, but in many cases it is possible to get excellent approximations to the optimal solution.

Integer programming is the analytical tool of choice in the formulation of such models. A very wide variety of non-convexities - all that can be thought of within the field of economic planning - can be represented or approximated adequately by integer programming models.[12/] In these models, some or all variables are restricted to integer values instead of being allowed to vary in a continuous fashion.

A survey of methods for solving integer programming problems is given in the Appendix. While exact solutions to such problems are often very difficult to obtain except for small problems, several methods exist that between them allow the generation of good sub-optimal solutions, together with upper bounds on the possibility of further improvement; thus the exact solution values can be approximated within a known margin of error.

The activity scales and resource allocations corresponding to the approximate solution will not necessarily be close to those corresponding to the exact optimum, since there are many cases in which widely divergent near-optimal solutions are known to exist.[13/]

In planning practice the knowledge of the exact optimum is seldom essential, for the following reasons :

11/ Contd. (transport, energy, communications) ; in housing and urban facilities; in government and other public services. See for example Rosenstein-Rodan (1943, 1961) Hirschman (1958).

12/ See Vietorisz (1964).

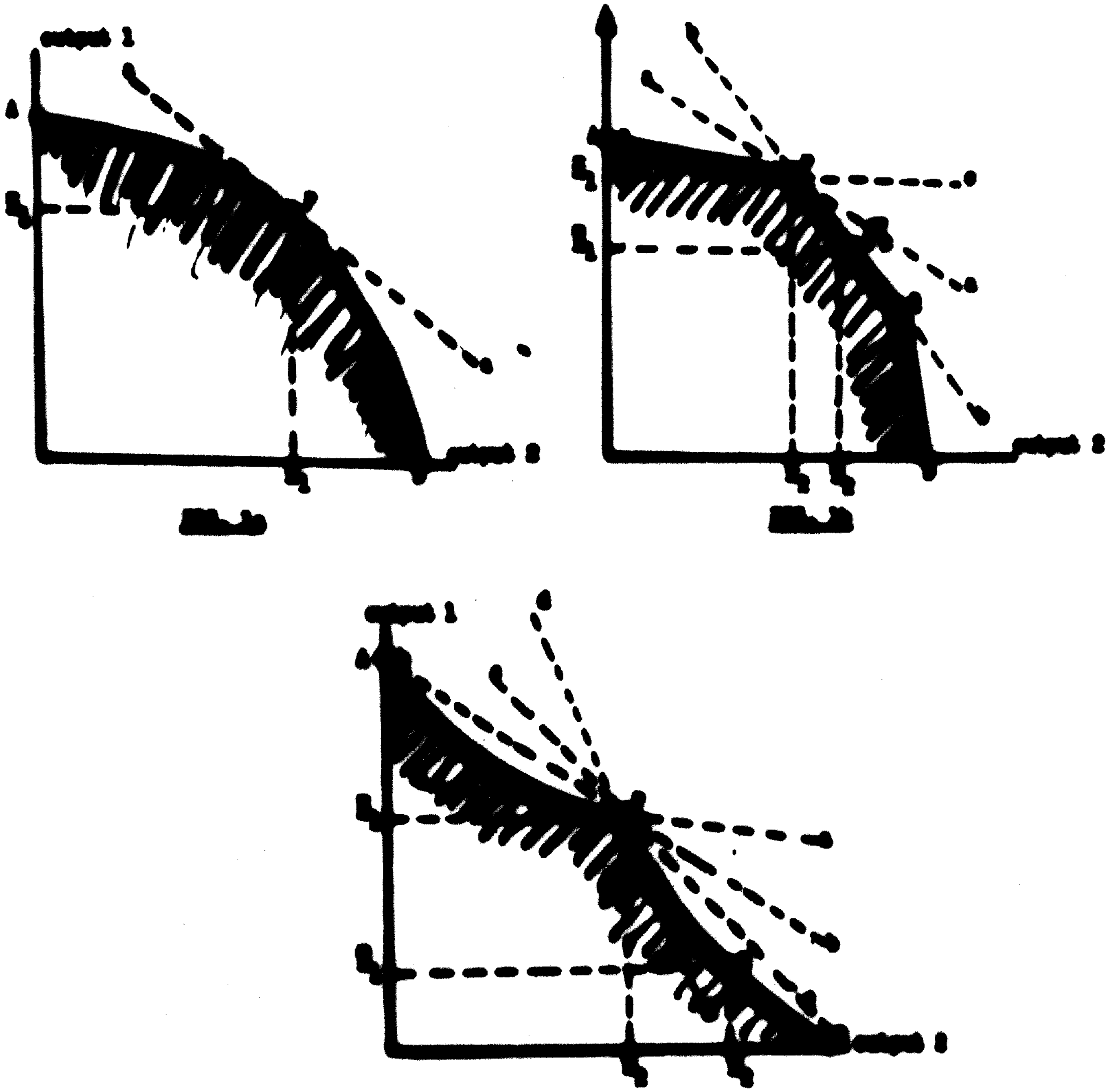
13/ See the tabulation of the best 100 solutions out of a total of 1024 enumerated combinations in Vietorisz (1964). A plot of the distribution of all 1024 solutions to the same problem is given in Vietorisz and Manne (1963).

1) optimising techniques are generally introduced into planning as an improvement over planning methods whose aim has been primarily the construction of consistent plans. Such methods aimed at consistency do give some attention to major overriding priorities but they do not carry out a systematic, iterative revision of all priorities such as characterizes any process of optimization. Whereas in solving a mathematical model the goal is to carry the iterative revisions to their logical conclusion, in the practice of plan preparation the number of revisions that can be actually carried out is necessarily limited by available personnel and time. Thus the goal is the more modest one of upgrading a feasible plan rather than the attainment of the exact optimum.

2) The data upon which the plan is based are subject to error; thus the exact optimum is also subject to error and only an optimal range of solutions can be specified with confidence.

3) The preferences of the decision makers can be described *ex ante* only in an approximate way, since final decisions always depend on a survey of meaningful available alternatives. Certain preferences, e.g. concerning the locational distribution of economic activities, may not even be discovered until a given plan that ignores these preferences is presented in detail. Therefore no single optimal solution is acceptable as the result of planning efforts, what is wanted is rather a range of alternative near-optimal solutions.

While an approximate solution to a nonconvex optimization problem is thus entirely acceptable, the large possible divergences between the activity scales and resource allocations of different near-optimal solutions do pose a problem in planning. This problem is related to the possibility of decentralization. In order to discuss this concept meaningfully, we have to explore the relationship between price-type and quantity-type control instruments in economic planning.


output 1
output 2

In <u>strictly convex</u>[14/] economic systems there is an exact correspondence between the optimal solutions that can be attained by means of price-type or by means of quantity-type control instruments (see Figure 1-a). If, however, the system contains <u>linear boundary segments</u> (as occurs in systems described by linear models. see Figure 1-b) this correspondence is destroyed: it is for example possible to specify output combinations that cannot be attained with certainty by means of price-type control instruments, only by means of quantitative controls.[15/] Whenever price-type control instruments fail to guarantee the attainment of desired constellations of inputs and/or outputs, decentralization based on the application of such instruments alone becomes impossible. Thus it is also impossible to define project evaluation criteria based on such price-type decentralizers, and it becomes necessary to resort to quantitative controls, or to a combination of price-type and quantitative controls.

14/ A point set S is strictly convex if $x_i \in S$, $i = 1, \ldots, r$ and $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$, implies that $(\sum_i \lambda_i x_i)$ is an interior point of S unless all x_i coincide. This cannot be true if the point set S has a linear boundary segment.

15/ For a detailed discussion, see Vietorisz (1965), Appendix 1. In reference to figure 1a, a combination of outputs such as point P on the production-possibility curve can be attained equally well by fixing a price ratio (line <u>aa</u>) or by fixing the quantities x_1 or x_2. Figure 1-b represents a system which is <u>convex with linear boundary segments</u>. A combination of outputs such as P or Q can still be attained equally well by price or quantity-type control instruments (e.g. : by fixing the price ratio anywhere between <u>bb</u> and <u>cc</u>, e.g. at <u>aa</u>; or by fixing x_1 or x_2; however, combinations exist such as R which cannot be unambiguously attained by means of price-type control instruments alone. Thus the price ratio <u>bb</u> will <u>sustain</u> point R in the sense that it will not initiate a movement away from point R, but this price ratio will <u>not assure</u> the attainment of point R <u>itself</u> in an optimizing solution, but only that of one of the infinite number of points along the segment PQ. In order to attain point R with certainty, the use of a quantity-type control instrument (fixing either x_1 or x_2) is indispensable.

The same conclusion applies in practice whenever the boundary of a system has segments that are indistinguishable from linear segments within the prevailing margin of error, and *a fortiori* when the boundary has nonconvex portions[16] (Figure 3-c.)

The full significance of these observations becomes apparent when their implications for multi-level planning are analysed. Effective decentralisation of information flow in economic planning requires that only essential decisions concerning the economy as a whole be taken explicitly at the top planning level, and that decisions of secondary detail be relegated to lower planning levels.[17] What happens in such multi-level planning structures when linear or nonconvex boundary segments are present at the lower levels? What effect do such boundary segments have on project evaluation? These are the principal questions that require further analysis.

16/ By reference to Figure 1-c which has nonconvex boundary segments, point P can be attained either by price-type or by quantity-type control instruments. Setting the price ratio anywhere between bb and cc will lead to point P from any other point along the boundary; widening this price range within the limits of aa and dd will still lead to point P from points in its own neighbourhood, although not necessarily from points near A or near B . Point P can also be attained by fixing either $\bar{x}_1$ or x_2 .

For the attainment of a point such as R , however, price-type control instruments become totally ineffective. Whereas in the case of a linear boundary it was at least possible to specify a price ratio (bb in Fig. 1-b) that would sustain point R , in the nonconvex case even this fails. A price ratio tangent to the boundary at R yields an unstable stationary point at R which corresponds not to the maximization of the value of output (as in the strictly convex or linear cases), but to a minimization of the same; the slightest movement away from R at this price ratio will initiate further cumulative movement towards P or B .

17/ Multi-level planning would be a practical necessity even if it were possible to obtain mathematical solutions to giant linear or integer programming models with many thousands of resources and activities. The reasons for this include the following. (1) Technical alternatives are hard to formulate explicitly over a sufficiently wide range of factor prices. (2) It is inefficient to formulate alternatives that will not be used; for this reason, the compilation of information and its analysis should alternate stage by stage. The

(Cont'd.)

B. The Decomposition Principle in Linear Systems

Some of the phenomena that occur in multi-level planning systems can be analysed by means of the "decomposition principle" developed originally for the solution of structured linear programming models.18/ Figure 2 indicates schematically the relationship between a two-level planning organization and the structure of a corresponding decomposition model. In the latter model nonzero technical coefficients occur only within the shaded blocks (Fig. 2a) and it can be seen that these co-efficients occur only within the shaded blocks (Fig. 2a) and it can be seen that these coefficients fall into two broad groups. First, there are the coefficients of the so-called "special resources" of each sector. The special resources of Sector 1 can have nonzero coefficients only in the projects of Sector 1, and likewise for the special resources of Sectors 2 and 3. Second, there are certain resources that may have nonzero coefficients in any sectoral project; these are designated as "connecting resources". It will be noted that in addition to sectoral projects that form the columns of the table there is also a column designated as "exogenous" (first column). While it is assumed that the scale at which each sectoral project can be carried out is variable, the scale of the

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17/ Cont'd. ..latter process can be carried out most effectively near the sources of technical information in individual sectors of the economy. (3) The structure of a large model cannot be intuitively grasped, and therefore its blind application is hazardous; this difficulty can be overcome by coordinating a number of smaller models. (4) Plan formulation must take into account the modes of execution : this requires familiarity with technical detail that is readily available only near the operating levels. (5) Plans have to be readjusted to changing circumstances in the course of execution. Many of these changes show up at or near the operating level; thus planning capability at lower levels facilitates efficient adjustment to such changes. For a discussion of some of these points see Clopper Almon (in Dantzig, 1963, pp. 462-465), and Vietorisz (1963:SSE).

18/ Dantzig and Wolfe (1961); see also Dantzig (1963), Gomory (1963:LNC), Kornai and Liptak (1965).

FIGURE 2

MULTI-LEVEL PLANNING AND STRUCTURE OF DECOMPOSITION MODELS

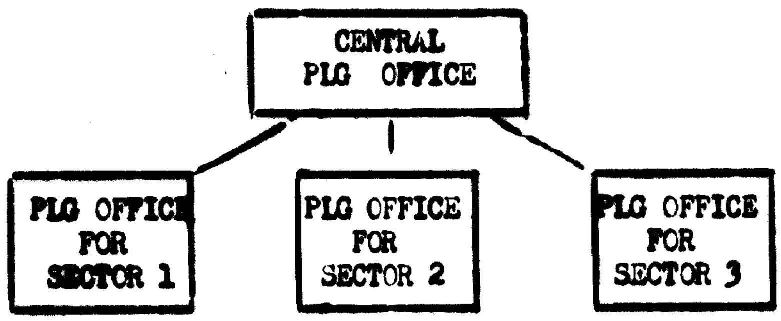

Fig. 2a: TWO-LEVEL PLANNING ORGANIZATION

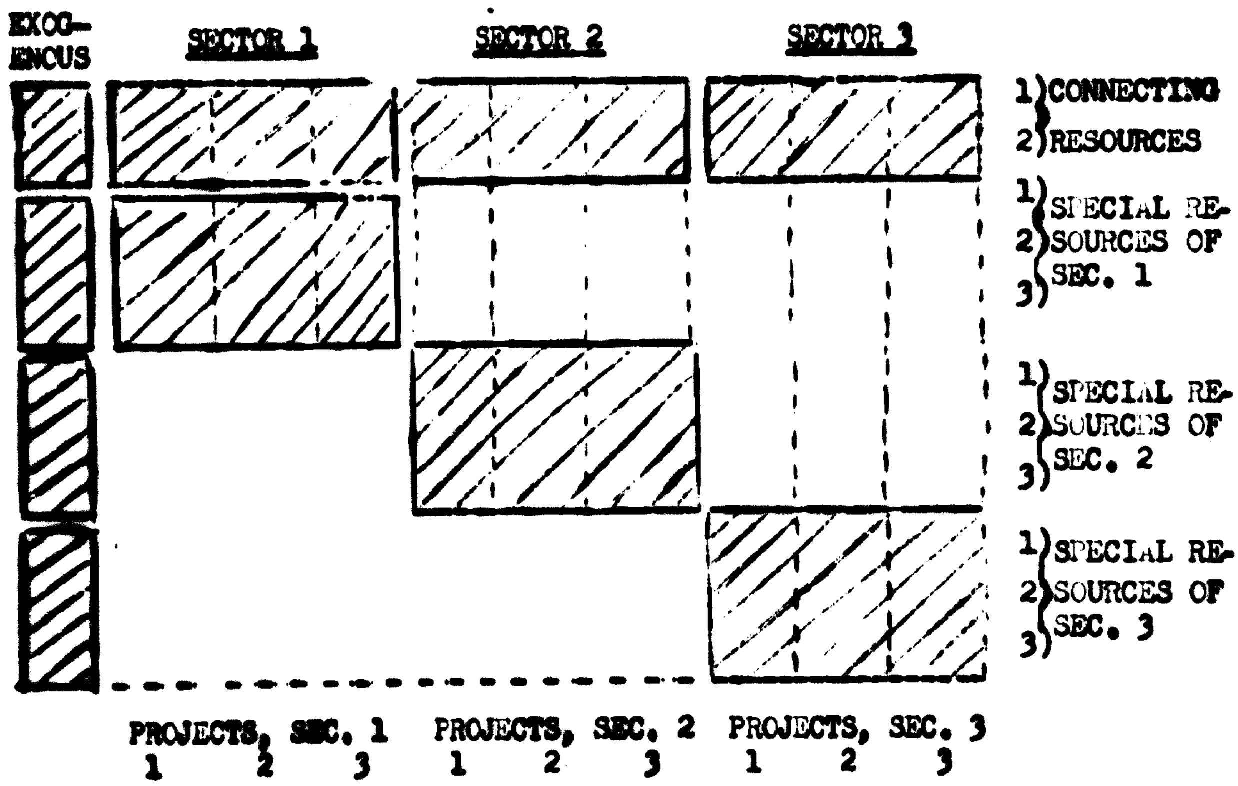

Fig. 2b: SPECIAL STRUCTURE OF DECOMPOSITION MODEL

exogenous column is fixed. This column usually contains the given total supplies and demands of each resource. The task is to find a "plan", i.e. a combination of project scales, that is consistent with the fixed resource supplies and demands, and that is in some sense efficient. Efficiency is defined in terms of maximizing the output or minimizing the input of a chosen connecting resource.

In such a structured model the consistency and efficiency-oriented decisions concerning the connecting resources correspond to the central planning level of a two-level planning organization, while the same kind of decisions concerning the special resources of the sectors correspond to the sectoral-level planning offices.

To what extent is it realistic to assume this special structure in decomposition models describing entire economies? The answer to this question hinges on the importance of direct interrelations between different sectors, manifested by coefficients of significant magnitude falling outside the shaded blocks in Figure 2a. It is known that when economies are described by input-output models these models can be arranged to an excellent degree of approximation in a block triangular form. The connecting resources of a decomposition model can thus be tentatively identified with the inputs of primary factors and with the inputs of resources (such as energy and transport) that occur near the base of the triangle of the rearranged input-output models; the remaining resources of the latter would then be treated as sectoral resources, with sectors delineated in such a way that interactions between sectors (other than via the connecting resources) be kept to a minimum. Such an approximation can be confidently assumed to be a reasonably good one for many economies; one may assume that corrections for direct interactions between sectors could then be undertaken by a few iterative revisions of the plans arrived at with the aid of the simplified description.19/

19/ The structure in Figure 2a is referred to technically as "angular"; it yields the simplest relationships between the connecting and the sectoral parts. The mathematics of block triangular systems has been explored by Dantzig (1963).

It should be noted that the decomposition structure described above is not the only approximation that can be applied to multi-level planning systems. While in this structure the resources subject to central decision (the connecting resources) and the special resources of the sectors form mutually exclusive classes, it is possible to define a system in which the resources subject to central decision are aggregated representations of the many detailed sectoral resources.[20] The logic of this kind of a system has been described qualitatively but has never been subjected to exact analysis.

Table 1 offers an illustrative numerical example of a decomposition model.[21] The model has two sectors with two special resources in each; and two connecting resources: capital and labor. There are four possible projects in each sector; the scales of these projects are variable and are designated by $X_1 \ldots X_4$ for sector 1, $X_5 \ldots X_8$ for sector 2. All numerical data obey the following sign convention: outputs or supplies are positive, inputs or demands are negative. Thus the capital and labor coefficients of all projects are negative (inputs); there are however exogenous supplies of these two factors, amounting to 350 units in the case of capital, and 2000 units in the case of labor. Once the scales of all projects are chosen in formulating a trial "plan", the flows of all resources are determined, and their balance can be verified. The difference between (1) all outputs and exogenous supplies of a resource (positive signs) and (2) all inputs and exogenous demands (negative signs) is defined as the surplus of the resource. If the surplus is zero, there is an exact balance; if positive, the resource is redundant; if negative, there is a bottleneck. In this problem,

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20/ UN-ECAFE (1961:FID), Chap. 2.

21/ The coefficients of this model have been based (with some necessary changes and additions) on a small illustrative model used by Chenery (1958:DPP), Table 2. Fixed-cost coefficients have been added; they are not used in the linear version of the model.

[illegible]

[illegible]

[illegible]		[illegible] →	MIN π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	
			[illegible] SECTOR 1					SECTOR 2				
CAPITAL [illegible]	σ_0	=	390?	-1.1	-1.25	-2	-2.5?	-1.0?	-2.5	-.6	-3.0	= Y_0 (=1)
LABOR	$0 \leq \sigma_1$	=	[illegible]	-12.5?	-7.5	-6.0	-7.0	-15.0?	-5.0	-4.0	-11.0	= Y_1
[illegible]. 1	$0 \leq \sigma_2$	=	-50	1	1	-.5	-.2					= Y_2
	$0 \leq \sigma_3$	=	-50		-.25	1	1					= Y_3
[illegible]. 2	$0 \leq \sigma_4$	=	-25					1	1	-.8		= Y_4
	$0 \leq \sigma_5$	=	-25					-.2	-.5	1	1	= Y_5 SHADOW PRICES
			X_0 (=1)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8 ACTIVITY SCALES	

[illegible] (nondegenerate?), SECTOR 1

A	X_1 and X_3	X_1 = 75.000	X_3 = 50.000	L = -1097.5?	K = -97.5
B	[illegible]	[illegible] = 86.715?	[illegible] = 71.429?	L = [illegible]	K = -[illegible].6
C	[illegible]	[illegible] = 80.000?	[illegible] = 20.000	L = -1200.0?	K = -[illegible].0
D	X_2 and X_4	X_2 = 63.[illegible]	X_4 = 65.789?	L = -[illegible].2	K = -[illegible].4

[illegible] (nondegenerate?), SECTOR 2

E	X_5 and X_7	X_5 = 53.572	X_7 = 35.724	L = -946.4?	K = -[illegible].3
H	X_6 and X_7	X_6 = 75.000	X_7 = 62.500	L = -[illegible].0	K = -[illegible].0
F	X_5 and X_8	X_5 = 25.000	X_8 = 30.000	L = -705.0?	K = -[illegible].0
G	X_6 and X_8	X_6 = 25.000	X_8 = 37.500	L = -787.5?	K = -137.5?

[illegible]

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
CAPITAL	-50	-5	-15	0	-10	0	-20	-5
LABOR	0	-50	0	0	0	-50	0	0

economy in the use of capital is chosen as the criterion of the efficiency of a plan; this is expressed by maximizing the surplus of capital. This formulation can be intuitively interpreted as follows: suppose 350 units represents the limit of capital stock that can be built up by saving and foreign borrowing; yet it is desired to cut back on the need for this saving and borrowing as much as possible. At the same time consistency of the plan requires meeting prescribed demands and keeping within available resource supplies; the latter conditions can be simply expressed as the avoidance of bottlenecks in any resource use.[22]

The model also determines the shadow prices of all resources. The price of capital is chosen as the "numeraire" resource whose price is set to unity and in terms of which all other prices will be expressed. The revenue (positive sign) or cost (negative sign) due to any resource can be determined once the shadow prices are given. the technical coefficients of a project are simply multiplied by these shadow prices. The difference between revenues and costs is the profit for any activity (variables in top margin). It is an interesting property of linear programming models that in solving for the most efficient set of project scales "X" that optimize the allocation of resources, a related "dual" problem of valuation of these same resources is also automatically solved. This problem consists in choosing shadow prices "Y" so as to minimize "profits" on the exogenous activity while profits on all projects are eliminated (as though these projects were in perfect competition). (See Mathematical Appendix).

The illustrative decomposition model of Table 1 is simple enough to permit a graphical representation by means of an Edgeworth box diagram. (See Figure 3.) In this diagram the total availabilities of the connecting resources (350 units of capital and 2000 units of labor) form the edges of the box. Resources used in each sector are measured

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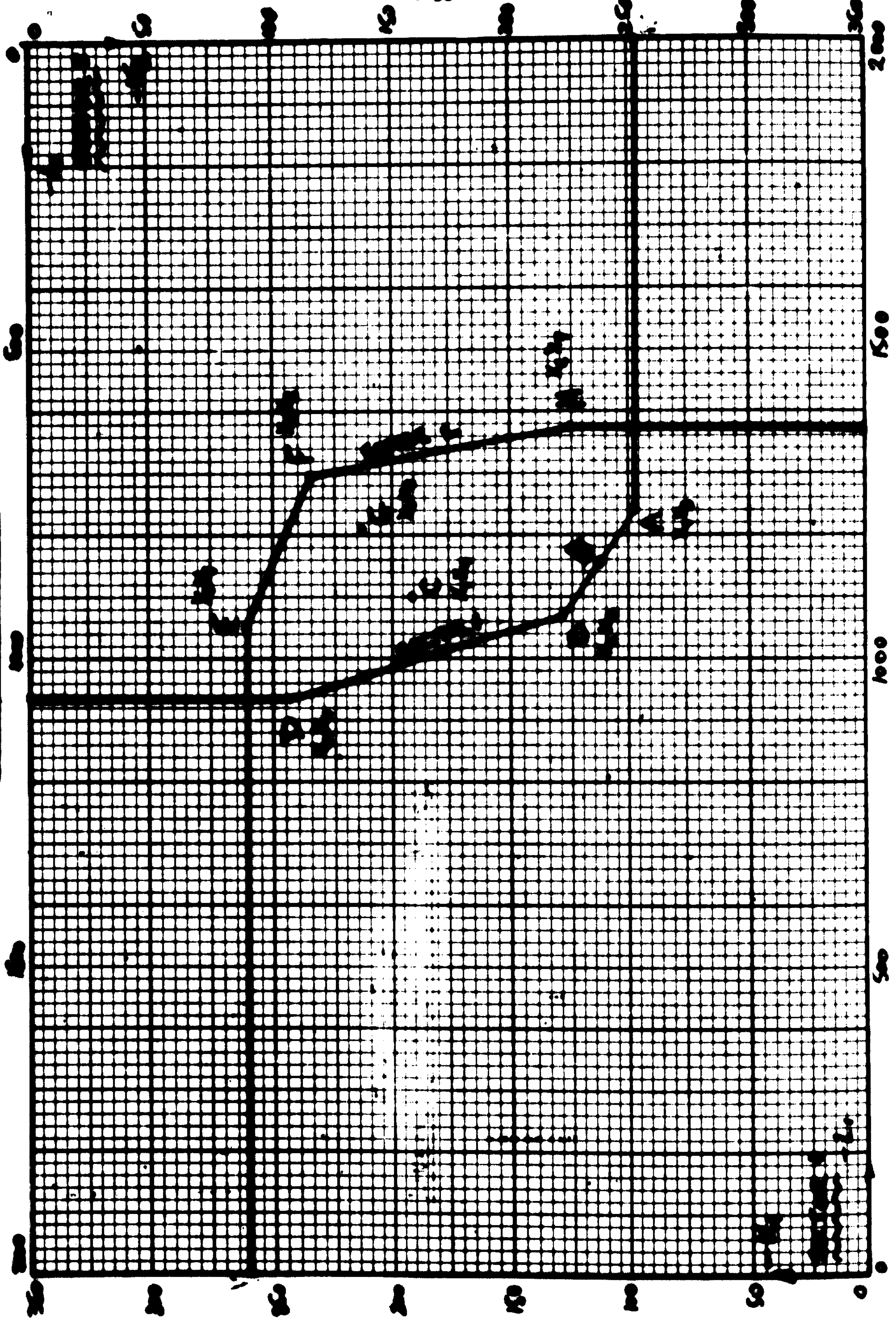
22/ For an interpretation of the system of Table 1 in ordinary algebraic equations, see Mathematical Appendix.



along the edges in opposite directions; thus any point in the diagram is a simultaneous representation of 4 variables: capital and labor used by Sector 1, and capital and labor used by Sector 2.

Points A, B, C, and D in the diagram represent four different complexes of projects that can be formed from the activities $X_1 \ldots X_4$ of Sector 1; points E, F, G, and H represent similar complexes formed from the activities of Sector 2. Each of these complexes contains two projects: this is the smallest number that permits satisfying the balances of the special resources in each sector.[23/] Table 1 contains a listing of the project scales and the total capital and labor requirements of each of these complexes; the respective project-scale variables are shown near each point in the graph. In preparing the graph in Figure 3, the <u>efficient</u> complexes of each sector have been connected by a line. Point C represents an inefficient complex in Sector 1 since it has larger requirements of <u>both</u> capital and labor than point B; thus it will never be attractive to use complex C. Likewise point G represents an inefficient complex in Sector 2.[24/]

The points along a line connecting two complexes, e.g. A and B, represent weighted averages of these two complexes. For example the midpoint of the AB line represents an average complex that is formed by running projects X_1 and X_3 of Complex A at half the scales shown in Table 1 ($X_1 = 37.5$; $X_2 = 25$); likewise running projects X_2 and X_3 of Complex B at half the scales shown for B in Table 1 ($X_2 = 42.858$, $X_3 = 35.715$); and summing the corresponding project scales (only X_2 requires

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23/ These complexes are extreme-point (vertex) solutions of the subproblems of Sectors 1 and 2. These subproblems are defined algebraically in the Mathematical Appendix and are discussed later in the text.

24/ Inefficient points need not use more capital and labor than any <u>one</u> point such as B or F; it is sufficient that they lie northeast (for Sec. 1) or Southwest (for Sec. 2) of the line connecting such complexes in any sector.

summation; thus $X_1 = 37.5$, $X_2 = 67.858$, $X_3 = 35.715$). It can be verified by simple algebra that the labor and capital inputs of the averaged complex fall exactly halfway between the labor and capital inputs of points A and B. In the present case the weighting was .5 and .5. Points other than the midpoint are obtained by using weights in different proportions. The weights may vary from 0 to 1 and they have to add up to unity. As long as this weighting rule is observed it is guaranteed that the special resource balances of each sector will be satisfied by the averaged complexes, even though the graph contains explicitly only the connecting factors. In addition to points lying on the connecting line between complexes such as A and B, the same guarantee applies also to any other point that can be attained starting with the former points and then disposing of (wasting, throwing away) some capital and/or labor.

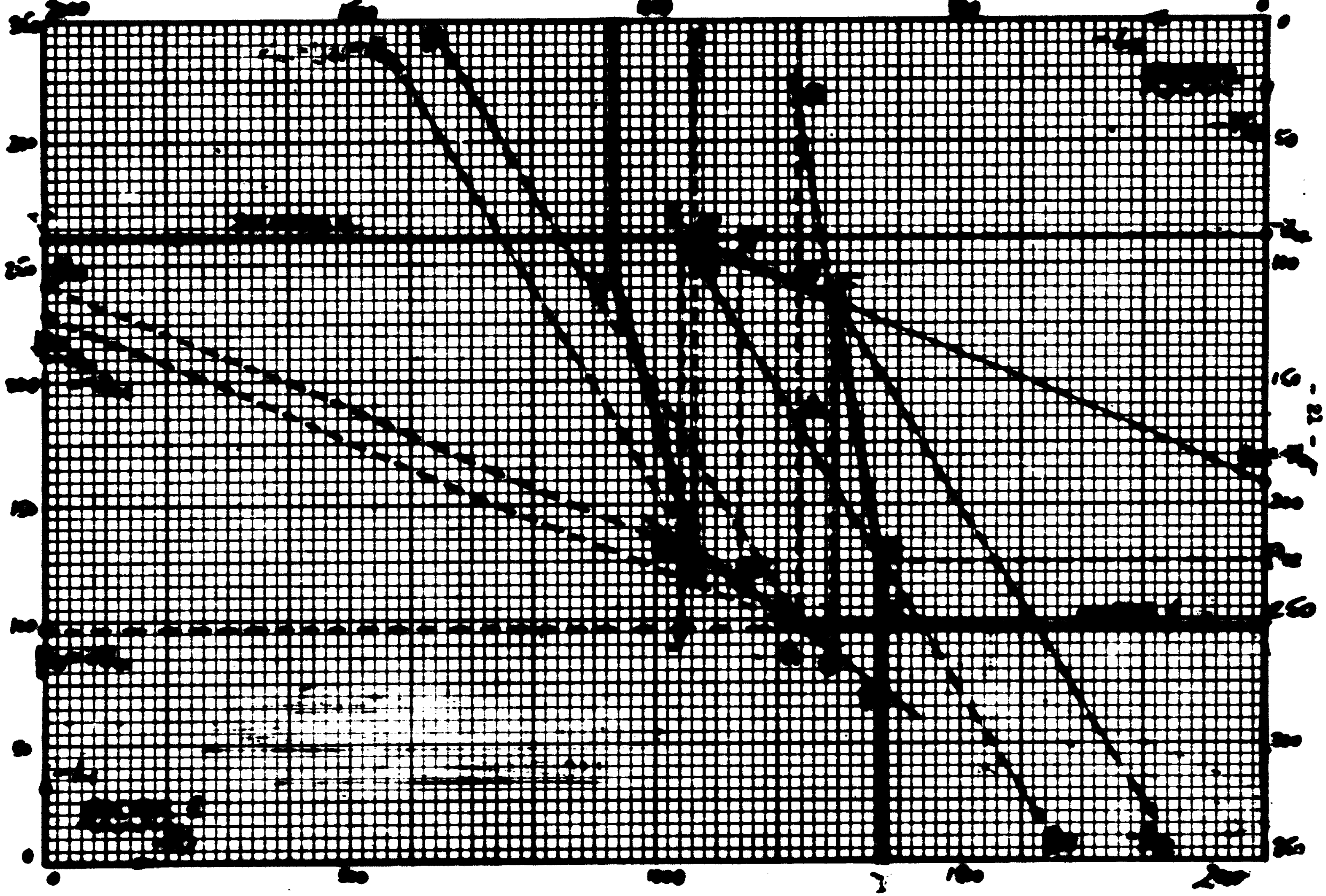
The two curves in Figure 3 can be regarded as generalized iso-product functions for the two sectors that describe the alternative combinations of the connecting factors (capital and labor) that can produce the given output of a sector. What is this "given output"? It cannot be identified with any single product since all special sectoral resources are on an equal footing and none can be regarded as "the" product of a sector; it is thus convenient to think of sectoral output as the entire task of satisfying the special resource balances.

The horizontal and vertical extensions of the two sectoral curves to the coordinate axes correspond to conventional useage in economics; they signify free disposal of redundant surpluses of the connecting factors, as mentioned above.

Figure 4 provides a graphical illustration of alternative methods available for finding an <u>optimal solution</u> to the model. Such a solution represents a <u>plan</u>, i.e., a set of projects with determined project scales, that is both <u>feasible</u> in the sense that it satisfies all resource balances, and efficient in the sense that it maximizes the surplus of capital (i.e., it minimizes capital requirements).

FIGURE 4

LINEAR [illegible] : GRAPHICAL SOLUTION



A "<u>feasible solution</u>" is a plan that satisfies all resource balances but is not necessarily optimal. Points B and T jointly represent such a plan. Point B on the iso-product line of Sector 1; thus it is sure to satisfy the balances of the special sectoral resources in this sector; point T is on the iso-product line of Sector 2 and thus satisfies the special resource balances of Sector 2. The labor requirements of the two points add up to 2000 and thus satisfy the labor balance. Accordingly all resource balances are satisfied and the plan is feasible. In order to determine whether it is also optimal, the capital requirements are identified: by inspection of Figure 4 they can be seen to leave a capital surplus exactly equal to the vertical distance BT. It remains to be decided whether other feasible solutions exist that leave a larger capital surplus.

Note that point B is one of the complexes of Sector 1 that has been presented in Table 1; while point T represents a weighted average of complexes E and F of Sector 2. This solution is labeled as "BEF" by reference to the sectoral complexes forming it. Table 2 (line 12) contains a listing of the quantitative characteristics of this solution including labor and capital requirements in each sector, capital surplus, and the weights used for averaging in each sector. In Sector 2 these weights are .926 and .074, respectively, for points E and F; in Sector 1, the weight is 1.000 for point B since this complex appears alone, without being averaged with another complex.

In general a feasible solution will be obtained when one point is selected from the iso-product line of each sector, attention being paid to joint labor requirements. When the two points fall on the same vertical line the joint labor requirements add up to 2000 units; when the point for Sector 1 falls to the left of the point for Sector 2 there will be an amount of redundant labor equal to the horizontal displacement between the two points (for example, when the combination AE is chosen); conversely when the point for Sector 1 falls to the <u>right</u> of the point for Sector 2 labor will be in a bottleneck condition (for example, combination BE). Since it is generally inefficient to leave labor

TABLE 2

LINEAR DECOMPOSITION MODEL: SELECTED SOLUTIONS

SOLUTION	FEAS-IBLE	λ_{11}(a)	λ_{12}(a)	AVG (1)	λ_{21}(b)	λ_{22}(b)	AVG (2)
$A\sigma_1 H$ (g)	YES	1	(137.5)[h]	I	1	-	H
$A\sigma_1 E$ (g)	NO	1	(183.9)[h]	J	1	-	E
$\sigma_1 EH$ (g)	NO	(NO POINT IN SECTOR 1: INFEASIBLE)					
AEH	YES	1	-	A	.428	.572	Λ
AFH	NO	1	-	A	1.719	-.719	M
AEF	YES	1	-	A	.238	.762	N
EFH	NO	(NO POINT IN SECTOR 1: INFEASIBLE)					
ABH	NO	1.828	-.828	P	1	-	H
ABE	NO	-.107	1.107	Q	1	-	E
BEH	YES	1	-	B	.945	.055	R
BFH	NO	1	-	B	3.795	-2.795	S
BEF	YES	1	-	B	.926	.074	T
ABF	NO	1.346	-.346	U	1	-	F

NOTES

(a) WEIGHTS FOR COMBINING COMPLEXES IN SECTOR 1
(b) SAME, SECTOR 2
(c) PRICE OF LABOR (PRICE OF CAPITAL $\cdot P_k \cdot$ Yo-1)
(d) LABOR REQUIREMENT (INHERENTLY NEG.)[k] IN SECTOR 1 AND 2
(e) SAME FOR CAPITAL
(f) SURPLUS OF CAPITAL (TO BE MAXIMIZED)
(g) σ_1 IN THE SOLUTION INDICATES A SURPLUS OF UNUSED LABOR
(h) THE NUMBER IN PARENTHESES IS THE VALUE OF σ_1

TABLE 2 (cont.)

SOLUTION	PL (c)	$-L_1$ (d)	$-L_2$ (d)	$-K_1$ (e)	$-K_2$ (e)	$\sigma_0 = 350 - K_1 - K_2$ (f)	
$A\sigma_1H$ (g)	0	1237.5	625	97.5	225.0	27.5	✓
$A\sigma_1E$ (g)	0	1237.5	946.4	97.5	89.3	163.2	
σ_1EH (g)	(NO POINT IN SECTOR 1: INFEASIBLE)						
AEH	.422	1237.5	762.5	97.5	166.9	85.6	✓
AFH	1.375	1237.5	762.5	97.5	35.9	216.6	
AEF	.106	1237.5	762.5	97.5	108.9	143.6	✓
EFH	(NO POINT IN SECTOR 1: INFEASIBLE						
ABH	.187	1375	625	71.7	225.0	53.3	
ABE	.187	1053.6	946.4	131.9	89.3	128.8	
BEH	.422	1071.4	928.6	128.6	96.8	124.6	✓
BFH	1.375	1071.4	928.6	128.6	-192.5	413.9	
BEF	.106	1071.4	928.6	128.6	91.2	130.2	✓
ABF	.187	1295.0	705.0	86.7	115.0	148.3	

NOTES

(a) WEIGHTS FOR COMBINING COMPLEXES IN SECTOR 1
(b) SAME, SECTOR 2
(c) PRICE OF LABOR (PRICE OF CAPITAL = P_K = Yo = 1)
(d) LABOR REQUIREMENT (INHERENTLY NEG.) IN SECTOR 1 AND 2
(e) SAME FOR CAPITAL
(f) SURPLUS OF CAPITAL (TO BE MAXIMIZED)
(g) σ_1 IN THE SOLUTION INDICATES A SURPLUS OF UNUSED LABOR
(h) THE NUMBER IN PARENTHESES IS THE VALUE OF σ_1

redundant, a convenient strategy for selecting feasible solutions in the course of optimization is to choose two points that lie on the intersection of a given vertical line with each of the two sectoral iso-product functions. The vertical distance between the two points measures the capital surplus corresponding to the given feasible solution. The geometric determination of the optimum is now obvious: it consists in selecting the vertical line that maximizes the distance between the two sectoral iso-product functions. In the present case the optimum is attained at AN, point N is a weighted average of complexes E and F in Sector 2. The solution, designated as A-F, will be found quantitatively described in the sixth line of Table 2.

This geometric method of finding a solution is not applicable to larger problems; Dantzig and Wolfe (1961) have however provided a generally applicable method which can also be followed by means of the graphical presentation in Figure 4. (See also Tables 2 and 3.)

Dantzig and Wolfe break down the overall problem into two parts: a "master problem" and "sectoral subproblems". (These correspond to central and sectoral-level planning decisions.) The master problem is formulated in terms of the connecting resources, in the present case labor and capital, and it is pieced together by averaging known sectoral complexes. The graph in Figure 4 represents this master problem. The master problem also determines prices for the connecting resources; in the present case, a price ratio for labor and capital. The sectoral subproblems, on the other hand, systematically find previously unknown sectoral complexes for inclusion in the master problem. The sectoral subproblems do not explicitly appear in the graph of Figure 4, but compliance with their balances is guaranteed by the averaging rules discussed above. The starting point of the technique has to be one known basic feasible solution to the master problem; given such a starting point, 25/

25/ If no basic feasible solution is known that would be suitable as a starting point, it is possible to construct one by algebraic techniques. See Dantzig and Wolfe (1961).

TABLE 3

LINEAR DECOMPOSITION MODEL: SOLUTION PATHS

N° OF SOL'N	SOL'N	P_1 (a)	$-q_1$ (b)	P_1+q_1 (c)	P_2	$-q_2$	P_2+q_2	VBL IN	SELEC-TION	VBL OUT	AVG' SOL'N	FEAS-IBLE
0	$A\sigma_1H$	97.5	97.5	0	225.0	89.3	135.7	– E	✓	$A\sigma_1$HE $A\sigma_1$HE $A\sigma_1$HF	– A J	✓
1	AEH	629.7	590.7	39.0	488.8	412.5	76.3	B F	✓	AEHF AEHF AEHF	– M N	✓
2	AEF	228.7	228.7	0	189.6	189.6	0	– –	OPT			
	ALTERNATE PATH IF COMPLEX "B" IS CHOSEN AS INCOMING VARIABLE IN SOLUTION 1 ABOVE											
1	AEH	(AS ABOVE)						B	✓	AEHB AEHB AEHB	R P Q	✓
3	BEH	590.7	590.7	0	488.8	412.5	76.3	– F	✓	BEHF BEHF BEHF	– S T	✓
4	BEF	242.2	228.7	13.5	189.6	189.6	0	– A	✓	BEFA BEFA BEFA	N U Q	✓
2	AEF	(AS ABOVE)							OPT			

NOTES

(a) "SUBCONTRACTING FEE" – – A REVENUE
(b) OPTIMAL COMBINED FACTOR COST ($z<0$)
(c) PROFIT ON OPTIMAL COMPLEX AT CURRENT PRICES

the interaction of the two parts of the problem guarantees the attainment of the optimal solution in a finite number of steps.

A <u>basic solution</u> contains the smallest number of nonzero variables that is compatible with the number of equations. In the master problem we have four equations (see Mathematical Appendix), one each for balancing capital and labor requirements, plus one for describing the averaging rules for complexes in each sector. The variables of the master problem are of two kinds: first, the weights to be applied to the individual complexes of each sector, and secondly, surpluses of labor and of capital that can also be interpreted as disposal activities. How many of these variables must be nonzero? Generally at least four.[26] One of these will be the capital surplus , which is being maximized; the other three may be three sectoral complexes, or two complexes and the labor surplus (disposal) activity, . In Figure 4 basic feasible solutions are obtained by selecting intersection points of a vertical line with the iso-product curves, as before, but with the additional restriction that the vertical line has to run through a vertex (a point for a single complex) in <u>one</u> of the sectors.[27] Solutions BEF and AEF that have been mentioned before are such basic solutions,

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26/ The number of variables including slacks (surpluses) in a linear programming problem exceeds the number of equations; the difference is known as the number of <u>degrees of freedom</u> of the system. A corresponding number of variables can be arbitrarily fixed, whereafter the values of the remaining variables are determined by solving the system of simultaneous equations. If the pre-set variables are assigned the value of zero we get a basic solution. In addition, by coincidence, the solution value of one or more of the variables that have <u>not</u> been pre-set may also turn out to be zero; in this case the number of <u>nonzero</u> variables will be <u>less</u> than the number of equations. Such a solution is termed "degenerate".

27/ Degenerate solutions are obtained when <u>by coincidence</u> complexes in both sectors fall on the same vertical line.

but solution ABEF corresponding to the vertical line VJ is not, since it contains <u>five</u> nonzero variables: capital surplus (the maximand), plus non-zero weights for each of the four complexes A and B in Sector 1, and E and F in Sector 2. In addition a solution such as AEH corresponding to the vertical line A is also a basic feasible solution, even though it is off the iso-product line of Sector 2, since the point λ can be obtained by averaging the two non-neighbouring complexes E and H. This point is of course not efficient since it could also be attained by starting with point N on the iso-product curve and then wasting some capital (corresponding to the distance Nλ).[28]

In the master subproblem not only the starting solution but all later solutions also have to be basic. The reason for this is that only basic solutions determine a unique price ratio for labor and capital which is needed in the sectoral subproblems. In a basic solution the price ratio is fixed by the slope of the averaging line segment that is intersected in one or the other of the two sectors. If the solution is nonbasic such as ABEF the vertical line VJ intersects line segments, generally of <u>different</u> slopes, in both sectors rather than passing through a vertex in one sector.

Let us now trace the course of optimization, using the Dantzig-Wolfe algorithm, by reference to Figure 4. Suppose the starting point is at the vertical line HI. This corresponds to a basic feasible solution (labeled "$AO_1'H$" in Table 3) in which complex A in Sector 1 and complex H in Sector 2 appear with unit weights; thus 2 weighting variables are nonzero. In addition there is some labor disposal: thus the labor surplus variable σ_1 will also be nonzero; its value

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[28] Basic solutions need not be feasible. If the solution value of any variable (a weight or a slack) turns out to be negative the solution is infeasible. In the graph of Figure 4 basic but infeasible solutionsare obtained if the vertical line is made to intersect not the line segment connecting two vertices but the continuation of such a line segment beyond one of the vertices. This represents an impermissible weighting of the two complexes, with one weight negative and the other exceeding unity. See for example point I corresponding to the averaging of complexes A and B in solution ABH (Table 2, line 8).

corresponds to the distance AI, which amounts to 137.5 units. The value of the maximand (the capital surplus variable [illegible]) corresponds to the distance AI, or 27.5 units.

We assume that at this point only complexes A and H are known. While in this problem there are in all only six efficient complexes, in larger problems the number of possible complexes increases combinatorially and thus at the beginning of the optimization there exists very little information concerning alternative efficient sectoral complexes. The task of the sectoral subproblems is precisely to identify previously unknown efficient sectoral complexes for inclusion in the master problem.

Looking at it another way, if all the efficient sectoral complexes were known from the very beginning the optimal solution to the master problem would immediately identify the optimal solution to the problem as a whole. Since, however, we are generally working with an incomplete list of complexes, we need a technique that will generate new complexes; and specifically, we have to generate those complexes that are needed for the optimal solution of the overall problem without having to enumerate all possible efficient sectoral complexes. We shall now indicate how the sectoral subproblems are utilized for achieving this aim.

In the starting solution the price ratio between labor and capital is determined by the slope of the line segment AI; in other words, the price of labor is zero. The price of capital is unity by assumption. Using these relative prices, the sectoral subproblems maximize the combined value of the connecting resources. In the present problem the connecting resources appear as inputs; thus we are in effect minimizing their combined cost. At the same time, the sectoral subproblems have to satisfy the balances of the special sectoral resources.

While in the graph of Figure 4 we do not show the special resource balances of the sectors in an explicit fashion, they are nevertheless allowed for by means of the averaging rules applicable to complexes. We know that the straight lines connecting the points corresponding to the sectoral complexes represent weighted averages of complexes; as long as the complexes themselves satisfy the special sectoral resource balances, these weighted averages will also do the same. In addition, we know that whenever we take one of the points corresponding to the complexes or their weighted averages and we subsequently dispose of (throw away) some labor or capital, we are still certain to satisfy the same sectoral balances. Thus we can map out <u>feasible areas</u> for both sectors in the graph: these consist of the iso-product lines <u>plus</u> all the points falling on the concave sides of these lines. Whenever a point is chosen within the feasible area of a given sector, it can thus be guaranteed that the special sectoral resource balances are satisfied. In this way we can use the graph of the master problem to represent possible solutions to the sectoral problems.

The question arises: in maximizing the combined value (minimizing the cost) of the connecting resources in the subproblems, using the price ratio of the starting solution, do we discover new complexes that are "more efficient" in some sense than the ones already known?

In the graph the combined value of the connecting resources is represented by <u>budget lines</u> whose slope equals the price ratio between labor and capital and whose intercept on the capital axis measures this combined value.[29/] The optimization in each sector is represented by a parallel shift of the budget line in such a way that the combined value of connecting resources is increased (combined cost is decreased), while maintaining at least one point of the budget line within the feasible area of the sector. In Sector 1 this procedure leads to point A which had already been known previously; but in Sector 2 the optimum

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29/ The budget line corresponds to the equation

$$P_L \cdot (-L) + P_K \cdot (-K) = (-z) ,$$

or:

$$(-K) = (-Z) - P_L \cdot (-L) ,$$

since $P_K = 1$. On the graph the axes correspond to (-K) and (-L); thus (-Z) is the intercept on the (-K) axis.

corresponds to a new complex E whose exact capital and labor requirements are disclosed by the optimization process.

In what sense can we assert that complex E is more efficient than previously known complexes?

In the starting solution complex H was the only known complex for Sector 2. The combined cost of the connecting resources for this complex can be read off by tracing a budget line with slope 0 to the capital axis of Sector 2: in the graph we read off 225 units at (this same value will also be found in Table 2 , in the line of solution 0 labeled "A.., H" , under p_2). 30/ The combined cost for complex E is however only slightly under 90 units as read off in the graph at (89.3 units under $-z_2$ in Table 3). Consequently the inclusion of complex E in the solution promises a combined cost improvement of 225.0–89.3 = 135.7 units , at the prevailing prices.

30/ p_2 is a shadow price in the master problem that corresponds to the equation describing the averaging rule for Sector 2. (See Mathematical Appendix.) Whenever a complex is included in a basic solution, i.e., when its weight is nonzero, the shadow profit for the column of this complex has to vanish. The mathematical reason for this is the well-known rule of complementary slacks applicable to linear programming problems; in economic terms the solution enforces perfect competition between all complexes included in it. Consequently the shadow price p_2 and the combined value of the connecting resources have to add up to zero; in other words the combined value equals $-p_2$.

p_2 can conveniently be interpreted as a "subcontracting fee." The master problem in effect places all complexes of a sector in competition with each other for the privilege of performing the task of the sector, namely satisfying the balances of the special sectoral resources. Whichever complex or complexes can perform this task at the lowest subcontracting fee will be selected to do the job. At any stage, the successful complexes will just break even: their combined cost for the connecting resources at the prevailing prices will just equal the subcontracting fee. The solution to the master problem can, however, be improved as long as sectoral optimization will disclose new complexes that can make a profit at the prevailing prices and prevailing subcontracting fees. When this is no longer possible, an overall optimum for the entire problem is attained.

In order to pass from the starting solution to the next solution of the master problem we will now want to include E in the solution. Since the solution is to be basic, however, we will have to drop some other complex or the labor surplus (disposal) activity. Table 3 indicates the three choices available for dropping variables and the corresponding solutions. (The capital surplus activity which is to be optimized is never dropped.) If we drop complex A we are left with no complex in Sector 1, and thus we have an infeasibility. If we drop [illegible] we get solution AEH which yields an average complex for Sector 2 at point λ, which is feasible. If we drop complex H we get solution A [illegible], E which leads to point J for Sector 1: an infeasible point, implying a negative [illegible]. (Numerical data describing each of these trial solutions will be found in Table 2.) Thus we have only one feasible choice, solution AEH. This is labeled as solution No. 1 in Table 3.

AEH determines a price ratio of .422 between labor and capital: this ratio equals the slope of the line connecting E and H. Budget lines with this slope yield new complexes in the course of the optimization in both sectoral subproblems: in Sector 1, the new complex is B, with a combined cost of connecting resources equal to $(-z_{11}) = 580.7$; while in Sector 2 the new complex is F with a combined cost of $(-z_{12}) = 488.8$. The cost improvement relative to solution AEH can be determined by comparison with the combined cost of A in Sector 1 which equals 580.7 (p_{11} in Figure 4; also in Table 3), and the combined cost of either E or H (these are equal) in Sector 2 which equals 412.5 (p_{12} in Figure 4, also in Table 3). The cost improvements are thus 39.0 and 76.3 units in Sectors 1 and 2, respectively.

Either one of these new complexes can be included in the solution of the master problem to get an improvement in the maximand [illegible]; it is, however, preferable to include the one with the larger cost improvement, namely F. Once again it becomes necessary to drop a variable from the solution in order to remain basic; the three choices are indicated in the line of Solution 1 in Table 3, and the resulting alternative solutions are numerically specified in Table 2. The only feasible choice is AEF. This solution determines a price ratio of 0.106 (equal to the slope of the segment EF); at this price ratio the budget lines disclose no new complexes in the course of the sectoral optimizations, and thus the solution AEF turns out to be optimal.

If at the stage of Solution 1 complex B had been included in the next solution rather than complex F, the path of optimization would have been slightly longer. In this case BEH turns out to be the next feasible solution; the price ratio remains .422 as in Solution 1. At this price ratio F is still present with a potential improvement and thus it is the next complex to be included in the master solution. The next feasible solution is obtained by dropping H; thus solution No. 4 is BEF, with a price ratio of 0.106. At this price ratio point A appears as an improved point in Sector 1; the next feasible solution, after dropping B, is AEF, the optimal solution.

From the point of view of project evaluation the significance of this analysis of the decomposition algorithm is that it discloses the fact of the insufficiency of price-type control instruments in attaining an optimal solution. As already discussed by Clopper Almon (in Dantzig, 1963, pp 462-465) the central planning office cannot guarantee the balance of connecting resources merely by setting the prices of these resources, since in a solution such as AEF the price ratio EF will not guarantee that Sector 1 will choose to produce exactly with the weighted average N of complexes E and F. Faced with the price ratio EF this sector may produce at any point along the segment EF, since all points along this segment are equally optimal at the stated price ratio, and there is no preference between them as far as Sector 2 alone is concerned. If the central planning office wants to make sure that the connecting resources will be adequately balanced it has to prescribe either a weighting of complexes E and F in Sector 2, or a quantitative allocation of labor and capital to this sector. At the same time, Sector 1 can be adequately regulated by the price ratio alone, since at the given price ratio it has a unique equilibrium position at A.

An interesting feature of the practical application of control instruments in this situation is that the central planning office will find it worth while to use both price and quantity-type control instruments, even though their joint use will be redundant in Sector 2.

> "They (the Central Trade office) announce in quantitative terms their feasible plan. They tell each plant manager how much of each traded commodity he must produce and how much he is allowed to purchase....They also announce the prices and direct that trade be conducted at these prices. They may also instruct the managers that, subject to their meeting the quantitative goals...they should also maximize profits. Such a rule is intended as a guide to avoid possible waste in the event that S (the quantitative goal) is not precisely achieved for one reason or another. It is important to note that they cannot tell the managers simply to maximize profits (omitting production goals, S) for if they did, Central Trade would almost certainly have difficulty with its constraints." 31/

At the project level, this insufficiency of price-type control instruments is translated into the insufficiency of the usual price-type project evaluation criteria, and calls attention to the fact that there is an inescapable minimum of quantitative control that has to be exercised even in highly decentralized systems. This does not mean, of course, that multi-level planning is useless; on the contrary, it reinforces the need for such planning, since it indicates that a decentralized market mechanism without a central decision making level will encounter the same indeterminacies that characterize the multi-level planning system with pure price-type coordination. Multi-level planning is at the same time preferable to pure central planning, since it results in an economy of information flow. It should be noted that the master problem in the decomposition algorithm requires no information on special sectoral resources and on particular sectoral projects or activities, it handles this information in an indirect fashion by means of delineating feasible regions for each sector on the basis of averaging known sectoral complexes.

31/ Almon in Dantzig (1963), 464-465. (Emphasis added).

The decomposition algorithm of Dantzig and Wolfe is not the only one that can be utilized for coordinating the master program with the sectoral subprograms. Kornai and Liptak (1965) have proposed a multi-level planning system in which the information flow is the reverse of that in a Dantzig-Wolfe system. In a Dantzig-Wolfe decomposition the master program signals prices to the sectoral subproblems and the latter signal combined utilizations of interconnecting resources by particular complexes to the master program; in other words, prices flow downward, quantities flow upward (except for the final quantitative control objectives fixed by the master program for the sectors in which averaging is required). In the Kornai-Liptak decomposition the master program passes allocations of the connecting resources to the individual sectors; the latter, in turn, signal their own sectoral shadow prices for these resources to the master program. Without going into detail concerning the Kornai-Liptak decomposition it can be seen by reference to Figure 4 that sectoral resource allocations of labor can be represented by a vertical line cutting the two iso-product curves; at any (basic or nonbasic) solution separate shadow prices can be determined for each sector. For an averaged complex, the shadow price coincides with the slope of the averaging segment; for a single complex (which appears with unit weight) the shadow price is distinct for increased and for decreased allocations. For non-optimal solutions the comparison of shadow prices for the two sectors will show an unambiguous difference; for example, for the basic solution BEF the shadow price of labor both in the upward and the downward direction is greater in Sector 1 than in Sector 2. This signals the need for increased labor allocation to Sector 1 at the expense of Sector 2. Conversely, for the basic solution A H an unambiguous price difference will exist in the opposite sense, signaling the need for increased labor allocation to Sector 1 at the expense of Sector 2. At the optimum, solution AEF, the vertical cut through A and N will yield a shadow price at N that is smaller than the shadow price at A for decreased labor allocation to Sector 1, and larger than the shadow price at A for increased labor allocation to Sector 1, thus signaling a stable equilibrium.

C. The Decomposition Principle in Nonconvex Systems.

The indeterminateness of control by means of purely price-type instruments that has been observed in convex systems with linear boundary segments will be present to an even stronger degree in systems that exhibit nonconvexities. In general, while in a linear system a given price ratio will sustain an optimum, in the sense that at this price ratio no movement away from the optimum will appear advantageous to any of the sectors (even though this optimum will not be attained without the intervention of quantitative controls), in a nonconvex system a set of prices will not even sustain the optimum in any stable sense.[32] In such a system there will be a constant tendency for some sectors to abandon the optimum position, and this tendency will have to be counter-acted by specific quantitative controls. The practical consequences of the introduction of such quantitative controls are not greatly different from the effects of such controls in systems with linear boundary segments; in this regard non-convexities merely reinforce the control requirements already manifest in the former systems. A more profound difference, however, concerns the applicability of iterative corrections for improving the efficiency of existing feasible solutions, since these tend to break down in the presence of nonconvexities.

We shall use the diagrammatic method developed for linear decompositions to indicate the changes that are introduced by considering the presence of nonconvexities. This will allow the application of some

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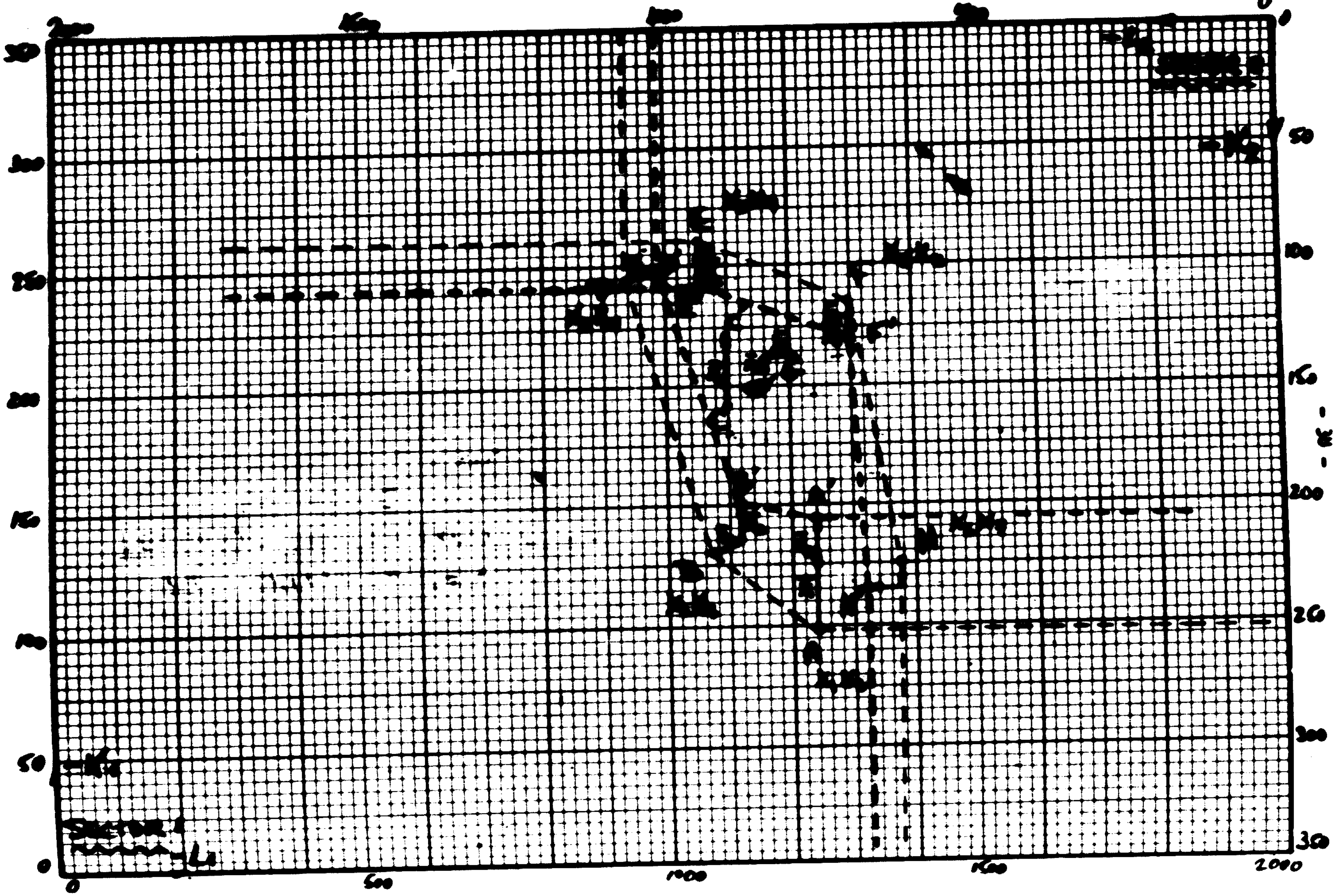
32/ More exactly, in a linear system small deviations from the optimal price ratio will set up only weak forces tending to move the sectors away from their previous positions, since the corresponding changes in the optimal value of the objective function are small; in non-convex systems, on the other hand, small changes of the price ratio can induce movements away from the previous position that are cumulative, since the farther the move has proceeded the stronger the incentive will generally be to move further still, as the difference in the value of the objective function at the previous position and at the end point of this cumulative movement can be very large.

judgements concerning the role these nonconvexities are likely to play in practical situations. One has the intuitive feeling that the presence of small nonconvexities cannot have a profoundly disturbing influence on the behaviour of largely convex systems, since common observation indicates that markets are often able to operate with reasonable efficiency in spite of the pervasive presence of fixed costs, economies of scale, and other nonconvexities. But what is "small" ? What systems are "largely convex" ? The diagrammatic method will offer some bases for judgement on these points.

Figure 5 indicates the first step in constructing a decomposition diagram with fixed costs included to represent nonconvexities. The fixed costs are expressed in terms of labor and capital requirements (see Table 1). For each complex such as A,B, etc., the fixed costs of the component projects (activities) are added up. In the graph of Figure 5, these additions are performed by means of vectors (arrows) which represent the labor and capital requirements of individual projects (activities). In this fashion, point A is carried into point A', point B into point B', etc. While points A,B,....in the diagrams have been referred to as "vertices" we shall refer to points A',B',.... as "apices" in order to keep the two kinds of points sharply distinguished.

Can apices be averaged? Generally not in a linear fashion, since for example, averaging apex A' and B' requires the joint use of projects X_1, X_2 and X_3 while apex A' allows only for the fixed cost of X_1 and X_3 and apex B' only for X_2 and X_3. Thus when two complexes are to be used jointly all the fixed costs of both complexes have to be incurred. Once all these fixed costs have been incurred, the variable costs can be averaged linearly as usual.[33] In Figure 6 these

33/ If fixed costs also comprise requirements of special sectoral resources these requirements can be translated into equivalent labor and capital requirements calculated at the marginal labor and capital requirements needed for producing the specified amounts of sectoral resources, on the assumption that all of these sectoral resources will in fact be produced in the optimal program, and that the corresponding fixed costs will thus be incurred in any event. This assumption may not be valid; and (Cont'd.)



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operations have been performed; for example, at A' the vector $\bar{x}_1$ has been added on, while at B' the vector $\bar{x}_2$ has been added on; the endpoints of the latter vectors can now be connected by a straight line. It is significant that the slope of this correct averaging line for apices A' and B' is the same as the slope of the vertex-to-vertex average. This is due to the fact that A, B, and the endpoints of the correct averaging line form a parallelogram, since the same three vectors have been added both to A and to B, even though in a different sequence. Thus the correct averaging line reflects marginal costs, while an apex-to-apex connecting line does not.

Two important qualifications to the foregoing procedure have to be noted:

(1) While the vertices C and G represent inefficient complexes in a linear system, it is by no means a foregone conclusion that they will also be inefficient in a nonconvex system comprising fixed costs. If, for example, the fixed costs associated with C were unusually small, it could easily happen that the correct averaging line involving C will pass in part on the infeasible side of the correct averaging lines for the other complexes, and will thus yield preferable points in this range.

(2) In a linear system averages of neighbouring efficient vertices are always superior to averages of non-neighbouring efficient vertices. In a nonconvex system with fixed costs this is not necessarily so; for example, the correct averages between apex A' and B' and between apex B' and D' may prove inferior in certain ranges to the correct average of apex A' and D' if the fixed costs associated with vertex B are unusually high.

Do the apices and the correct averaging lines appearing in Figure 6 jointly form iso-product lines for the two sectors? In answering

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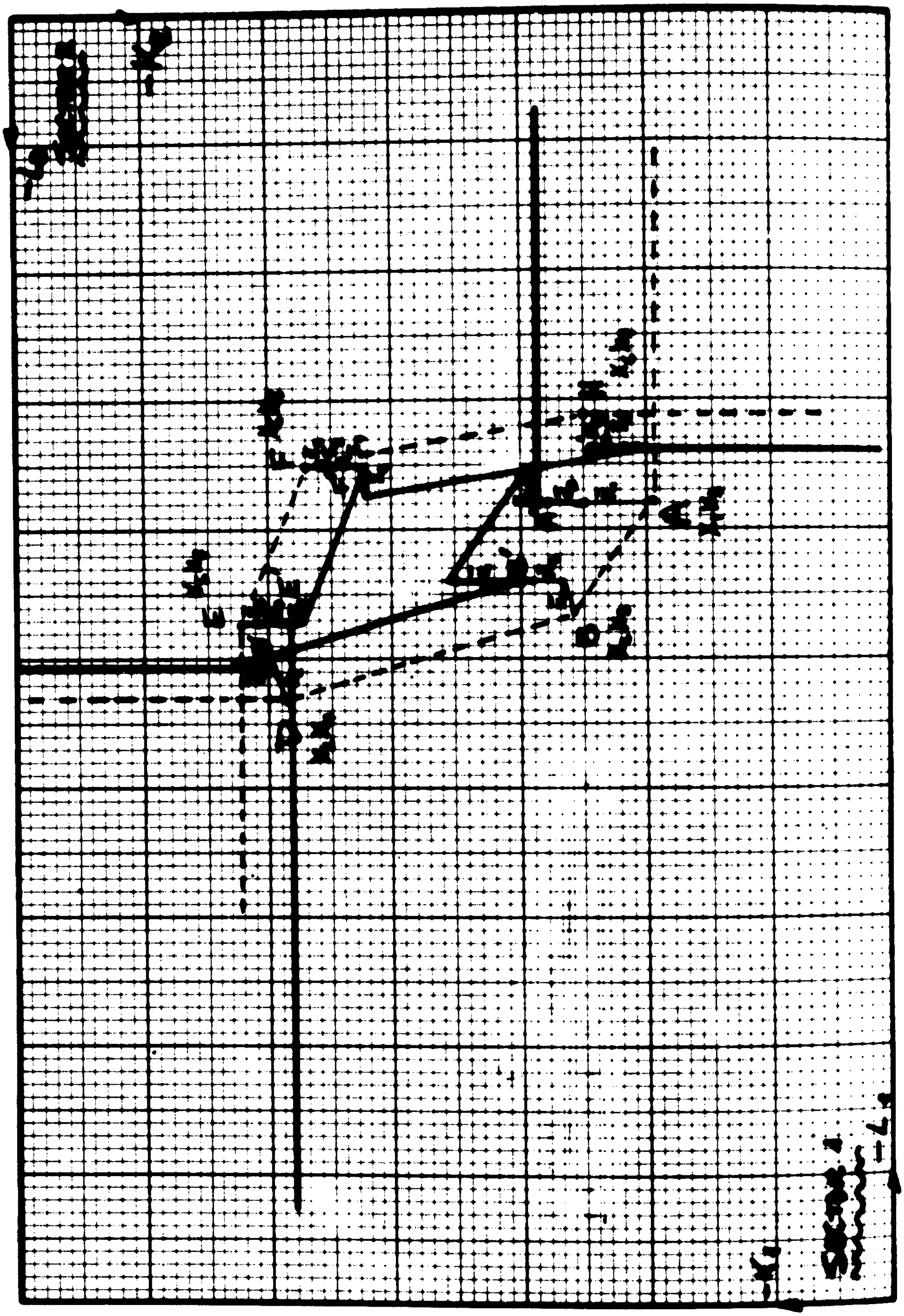
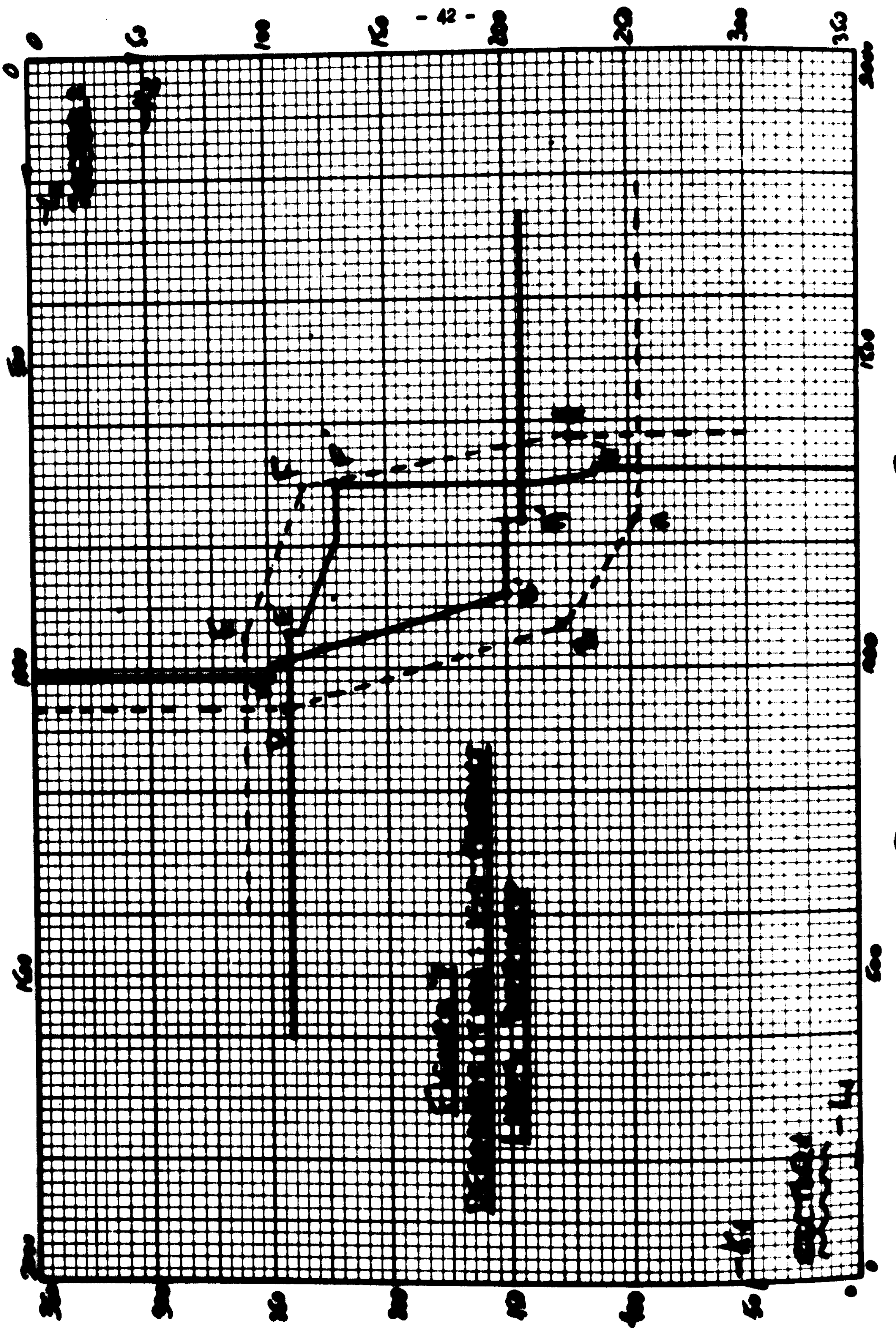
33/ Cont'd. ...there might exist some choice in the selection of activities for producing these fixed-cost components. We shall abstract from all of these secondary complications in the course of the present discussion.



this question it has to be remembered that free labor and capital disposal is at all times permitted; thus any point in the diagram representing a legitimate apex or average will dominate all points derivable from it by such disposal activities. Therefore B' will dominate all points on the correct averaging line between A' and B' that are to the Northeast of B'; and likewise for A'. As a result, the entire line connecting the endpoints of vectors $\bar{x}_1$ added to B' and $\bar{x}_2$ added to A' will disappear and will be replaced by a step function between A' and B' (see Figure 7). Applying the same considerations of dominance to other areas of the diagram we wind up with the iso-product lines of Figure 7 that have a much simpler configuration than the apices and correct averaging lines of Figure 6. This simplification of the diagram is not a special feature of the numerical example under study but a general phenomenon that is due to the fact that the correct averaging lines have pronounced dips at the apices where one fixed cost is in all cases eliminated. As a result the straight line segments representing variable costs are generally truncated near the apices and in some cases (as between A' and B') completely eliminated in favor of simple step functions.

What can be said about the nonconvex decomposition problem represented by the iso-product lines of Figure 7? In general when the lines are correctly drawn and all the apices that contribute specified ranges to the line of a sector are known it is possible to find a solution to the master problem without the need for considering all the detailed information represented by the specific sectoral resource balances and sectoral projects. A knowledge of the capital and labor requirements at these apices, together with correct averaging procedures is sufficient to guarantee an exact solution to the master problem. The averaging procedure in the present case can be based on a listing of projects included in each complex together with their fixed capital and labor requirements; when two or more complexes are averaged, it is then simply necessary to check off all projects that are included and to add up their fixed costs. Formally, the master problem becomes an integer



programming problem in which the averaging of the variable costs of the complexes is conditional on incurring all the requisite fixed costs.

In practical applications the shortcoming of this procedure is twofold. First, it is difficult to solve the integer programming master problem; second, the availability of information concerning the requisite apices can by no means be taken for granted, since the number of such apices increases combinatorially with the size of the problem. As discussed in connection with the linear decomposition problem, the virtue of the Dantzig-Wolfe algorithm is precisely that it generates new complexes as they are needed, thereby shortcutting the enumeration of efficient complexes. The question is, can a similar procedure be developed for the nonconvex case?

No such procedure is presently available and the difficulties standing in the way of evolving one are great.

(1) To begin with, the meaning of prices in the master problem -- as in integer programming problems in general -- now becomes ambiguous. In Figure 8 the overall optimum happens to be at the vertical line passing through B', as can be verified geometrically or by means of a simple computation. What is the proper price ratio between labor and capital characterizing this optimum? Is it the slope of the iso-product line at J ? This slope, as noted before, corresponds to the averaging of variable costs, i.e., to the slope of the line EF; it is thus a marginal cost ratio. Or is the proper price ratio the slope of the apex-to-apex connecting line, E'F' ? In the present case the two slopes are not greatly different; but with only a small change in some of the fixed costs the optimum can be shifted to a vertical line such as MN. Here we have three possible price ratios: the former two, and in addition, the zero labor price corresponding to labor disposal.

(2) Next, we have to ask what the role of such a price ratio is going to be. Will it be used, as in the linear decomposition problem,

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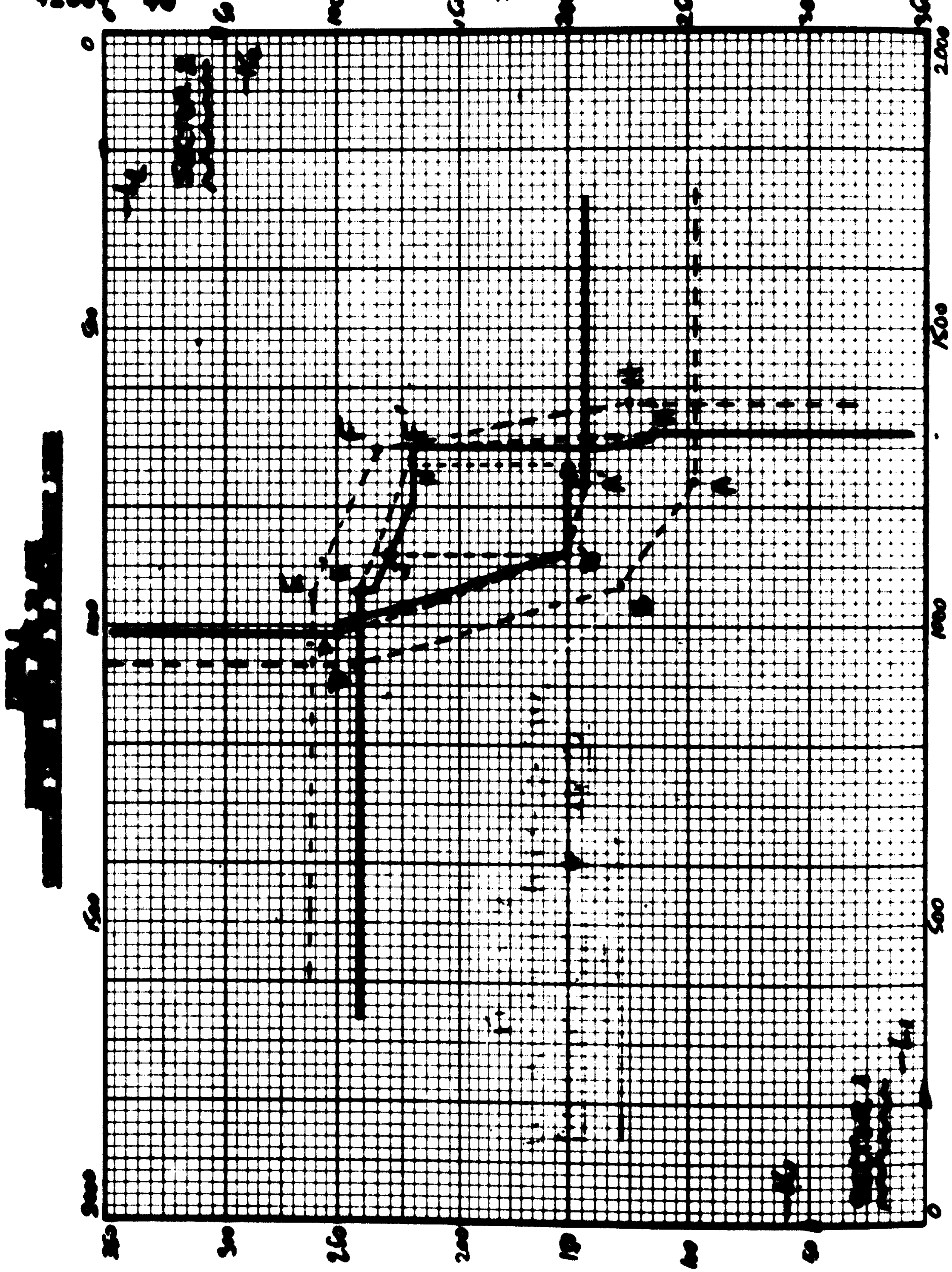


in a search for new efficient complexes ? If so, the sectoral subproblems become integer programming problems involving the minimization of combined costs (as in the linear case), but with allowance for fixed costs of the individual projects. In the present illustrative case (see Figure 8) such sectoral optimizations performed at the proper price ratios will identify all apices that participate in defining the iso-product lines; however, this cannot be generally guaranteed, because apices can also occur in local indentations of the iso-product lines that will not be optimal under any price ratio. Alternately, the role of the price ratio may be taken to be to sustain an optimum, as in the linear case; if so, the local marginal price is the proper one to use, but (as noted before) such a price will sustain the optimum only in a most unstable way, since the slightest change in the price ratio will generally precipitate a cumulative movement away from the optimum.

On the basis of considerations such as these it is clear that the concept of a unique price ratio characterizing convex systems cannot be extended to nonconvex systems; it rather appears necessary to define different price concepts for serving different kinds of functions. The price concept needed for identifying new complexes for inclusion in the master problem is an apex-to-apex connecting price, while the price concept needed for sustaining an optimum, if only in an unstable manner, is a marginal price. As in the linear case, quantitative control instruments are needed for making sure that the system arrives at an optimum and, in the nonconvex case, also that it remains there. The role of marginal prices in such a situation can be the one of allowing for small corrections in case of unforeseen deviations from optimal quantities in the course of plan execution, in a way analogous to the linear case discussed earlier.

Figures 8 and 9 have been drawn to indicate two possible approximations to the derivation of an exact optimum in such nonconvex decomposition problems.

(1) In Figure 8 the apex-to-apex connecting lines are shown in relation to the correct iso-product lines. It can be seen that the apex-to-apex connecting lines yield a linear approximation to the nonconvex master problem while they maintain the nonconvex nature of the sectoral problems. An approximate overall solution can be obtained in an iterative fashion by determining successive price ratios from the basic solutions of the linearized master problem; these price ratios are then applied to sectoral integer programming problems that will identify new efficient complexes if such are available. The latter are included in the linearized master problem and the procedure is iterated. This approximation has the virtue of generating new complexes only as needed, similarly to a linear decomposition problem.

What will be the characteristics of this approximation?

(a) It will always yield an overestimate of the optimal value of the objective function, since it ignores regions of local indentation between apices. These apices are averaged in a simple linear fashion, ignoring the correct averaging procedures.

(b) The approximation will be a good one to the extent that nonconvexities are weak; i.e. to the extent that local indentations are small in comparison with changes of capital surplus corresponding to different basic solutions of the linearized master problem. In other words, the apex-to-apex connecting line stays close to the true iso-product line, where closeness is measured in reference to a feasible area that is convex in the large and has only small local nonconvexities. Note that the graphical representation permits an intuitive appraisal of the relative roles played by convexity-in-the-large versus nonconvexity-in-the-small.

(c) Such a situation is likely to arise when either fixed costs are small in relation to the changes of variable cost over the averaging ranges, or else, where the fixed costs of many common projects are shared between neighbouring complexes that differ only slightly in project composition.

(d) Another important situation of this kind arises when fixed costs in a sector are incurred stepwise; in other words, when projects with given fixed costs are limited to a maximum scale, beyond which the fixed cost has to be duplicated. This has the effect of reducing the size of the abrupt increase in correct averaged costs near the apices and brings the iso-product line within a fraction of the distance from the apex-to-apex connecting line that prevails when fixed costs have to be incurred in a single step. This case corresponds closely to the classical case of the equilibrium of many small firms, each with its own fixed costs.

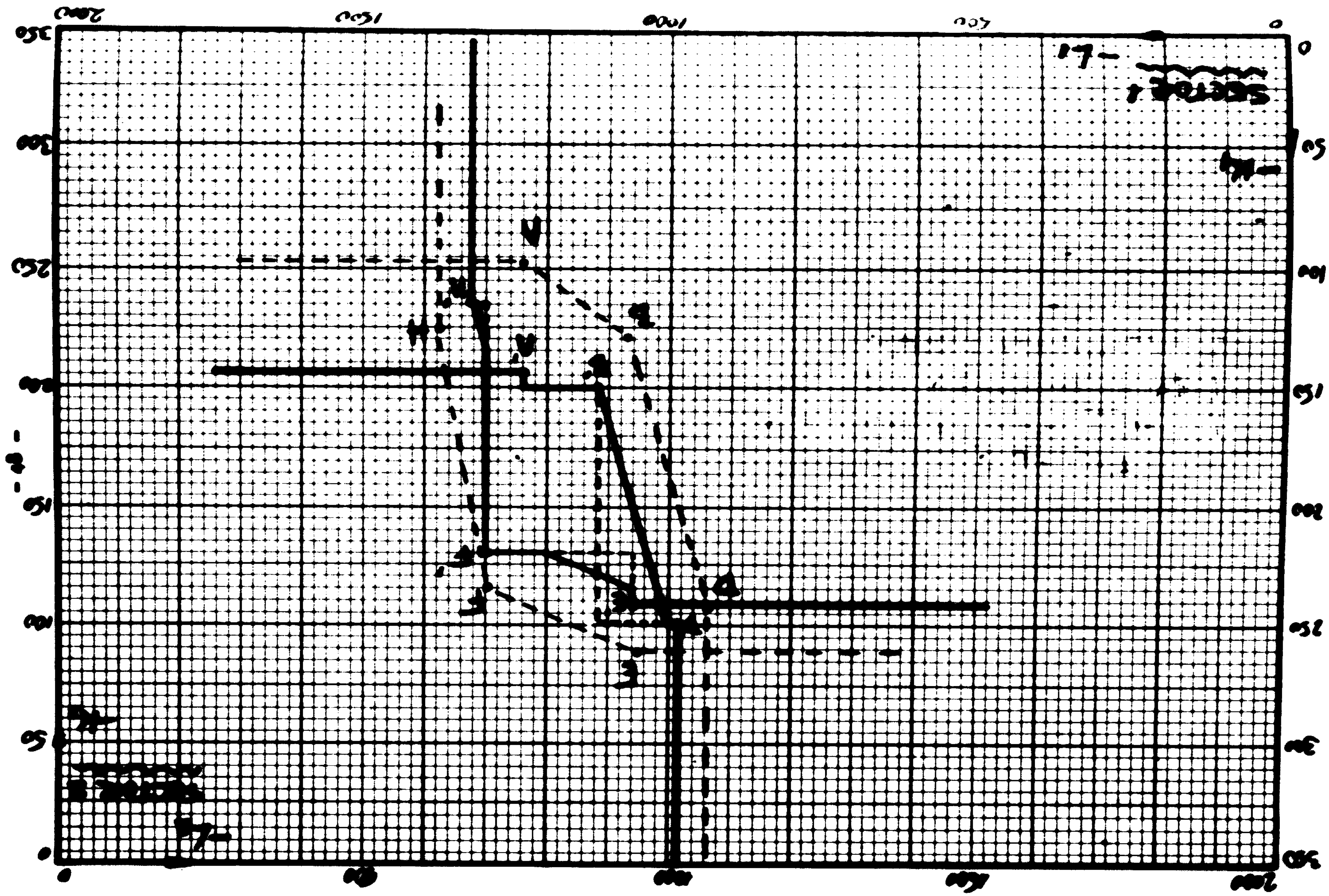
(e) The computation will be efficient to the extent that the sectoral integer programming problems are of small size or have a special structure that renders them easy to manage.

(2) In Figure 9 the unaveraged apices are shown in relation to the correct iso-product line. An approximation to the iso-product line can be pieced together from these apices by adding vertical and horizontal extensions corresponding to free labor and capital disposal activities. In other words, whereas in the previously discussed approximation we formed apex-to-apex connecting lines that acted as though the apices could be averaged in a straight linear fashion, the present approximation discards the tool of averaging altogether and simply disposes of labor and capital not required by one apex or another in a given solution. As a result, solutions are restricted to one complex in each sector.

The characteristics of this approximation are the following.

(a) It always yields an <u>underestimate</u> of the optimal value of the objective function, for two reasons: first, because it ignores the possibility of legitimate averaging; secondly, because it generally operates with an incomplete list of apices if the problem is large.

(b) The master problem is now an integer programming problem which does not yield useful prices for defining sectoral objective functions.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(c) Individual apices may be generated in any convenient way; for example, by means of simultaneously undertaking the first kind of approximation (apex-to-apex connecting lines).

(d) This approximation will be a good one whenever nonconvexities are large in relation to changes in the objective function (capital surplus) corresponding to widely separated solutions; in other words, when the sectors are characterized by a few major indivisibilities. The reason for this is, first, that in the case of large nonconvexities not much is lost by refraining from averaging; secondly, that if nonconvexities are large, the number of apices that contribute to the correct iso-product line in any sector is necessarily small and thus the apices are relatively easy to identify on the basis of empirical considerations that are likely to be well known to the planners familiar with the sector; thus the possibility of missing significant apices is greatly reduced.

(e) The computation will be efficient to the extent that the master integer programming problem is of manageable size.

In sum, the two approximations are complementary. Between them, they yield both an upper and a lower bound on the value of the optimal solution; in addition, each tends to be close in cases with opposite characteristics. The first approximation tends to be close when the feasible area within a sector is convex in the large and has only small local non-convexities; while the second approximation tends to be close when a sector is characterized by a few major indivisibilities.[34]

34/ In any given practical problem it appears advantageous to undertake both approximations simultaneously. It might perhaps also be possible to combine thw two approximations, choosing the better approximation to represent any given sector; in this case, however, the bounding properties of the separate approximations would be lost and the exact nature of the iterative algorithm would be placed in doubt.

It is noteworthy that presently available practical methods of coping with nonconvexities in economics tend to run in the direction of these two approximations: thus in the case of small nonconvexities an attempt is made to define some reasonable average cost and price that will take into account the presence of fixed costs, while in the case of major indivisibilities the operation of the price system is invoked only after quantitative decisions have been taken in regard to these indivisibilities on other than pricing criteria.

[illegible]

ANNEX I

Solving Integer Programming Problems: A Survey

Several algorithms exist for obtaining optimal solutions to integer programming problems. 1/ These algorithms are known to terminate in a finite number of steps, but when the number of variables is large, "finite" may mean a very large number indeed, since the number of alternatives that can conceivably be touched upon in the course of a solution, rises combinationally with the number of integer variables. Thus, mixed problems with as few as ten integer variables may not terminate in as many as several thousand iterations. 2/ In this respect, integer programming is completely different from linear programming where it has been found empirically (although no proof exists) that problems terminate in approximately three times as many iterations as there are constraints. With integer programming problems, it has been found that the standard cutting-plane type algorithms work with a quite different efficiency on problems with identical construction but randomly chosen parameters: some run well, some take quite a bit longer, and some make very poor progress. A recent series of experiments has demonstrated, moreover, that attempts to improve the amount of progress from iteration to iteration do not generally lead to faster overall progress. 3/

When these integer programming algorithms do not terminate due to practical inability of carrying the number of iterations beyond a reasonable limit even on the largest electronic computers, they at least yield a bound on the optimal solution in the sense that they indicate a point beyond which no improvement can ever be carried. Unfortunately, the published algorithms are of the "dual" type: they proceed by the way of trial solutions each of which is "dual feasible". Dual feasibility means that individual activities are always maintained in perfect priority ordering, in the sense that they either break even or show losses at the existing shadow prices, but never profits; at the same time, they are not "primal feasible"

1/ See Dantzig (1963), Chap. 26; Gomory (1963, 1965). For a discussion, see Vietorisz (1964: IDP)

2/ Vietorisz (1964:IDP)

3/ Work in progress

except at the optimal solution itself, i.e., they do not balance out all resources: some may be overdrawn (in a bottleneck condition). Thus, if the algorithm is broken off before it terminates, it will not yield simply a sub-optimal solution, but a hyper-optimal one: a "solution" that is "too good" as it leaves some of the constraints unsatisfied. Thus, such a solution can be used only as a bound: it shows that no primal-feasible solution can be better than the break-off solution, but it gives no clue to what would be a good primal-feasible solution. The latter has to be generated independently by one of the techniques to be discussed below. Still, the bound is highly useful because if any primal-feasible solution is known, the bound indicates how much room for potential further improvement is still available.

A new type of algorithm has been published recently 4/ that is far more efficient for certain classes of problems: optimal solutions are attained in a limited number of steps or, if the problem turns out not be have been of the proper type, this algorithm also yields a bound on the optimal solution.

It should be noted that ordinary linear programming (with integer restrictions dropped) always yields an upper bound and (by rounding) a sub-optimal solution and lower bound as well, but these bounds generally tend to leave a wide range of uncertainty.

Some mixed integer algorithms do generate primal-feasible solutions 5/. These algorithms, if they fail to terminate, give simultaneous upper and lower bounds on the value of the optimal solution. Sub-optimal solutions giving lower bounds on the value of the optimum may also be obtained by one of the following devices:

1. Enumeration of individual integer combinations, particularly in the case of zero-one type integer variables. If the number of variables is small, the enumeration can be complete, and the optimum

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4/ Gomory (1965)

5/ Algorithms by R.E. Gomory cited in Vietorisz (1964:IDP).

can be selected by inspection; if the number of combinations is too large to be exhausted in this way, certain types of combinations can still be enumerated. For example, in the case of locational problems involving several processes, the total number of plant combinations, particularly in multi-period models, can easily become unmanageable but integrated plants at individual location can still often be enumerated. Other selected combinations can also be investigated one by one. The best of the enumerated solutions can then be accepted as a sub-optimal solution yielding a lower bound on the optimal solution. 6/

2. Steepest ascent and other gradient methods. 7/ These treat the integer programming problem by a method similar to gradient methods applicable to convex problems. This method will always lead to a local optimum. In the case of convex problems a local optimum must also be an overall optimum while in the case of non-convex problems, several local optima may exist. Gradient methods will lead to one or another of these optima according to where the starting point of the procedure is chosen. Thus by attempting the ascent from several widely separated starting points, the chances of hitting the overall optimum are improved.

3. In the case of optimizing an objective function with a non-convex preference set, subject to linear constraints, an interesting trial and error algorithm is available 8/ that starts with a local optimum and explores basic solutions systematically around this local optimum for possible improvements in the objective function, thereby yielding a better probability of attaining the overall optimum.

By choosing a mixture of the approaches noted above, almost any integer programming problem can be tackled with a fair confidence of establishing narrow upper and lower bounds on the solution. If these are within error limits, the problem can, in fact, be taken as solved.

6/ See Victorisz and Manne (1963)

7/ For a survey, see Victorisz (1964:IDP)

8/ Kornai et al, "Mathematical Programming of the Development of Hungarian Synthetic Fiber Production" (in Hungarian), Centre for Computing Techniques, Hungarian Academy of Sciences, Budapest, 1963, 190 pp. (mimeog).

A further strategy, also based on error limits, is the following: the parameters of the problem can be subjected to random variations within their own error limits, in the hope of generating a parameter combination that is equivalent in practice to the original formulation yet may show improved convergence characteristics.[9]

9/ Work in progress.

ANNEX II

Mathematical Appendix

The problem stated in Table I follows the condensed format of Tucker (in Graves and Wolfe, 1963). A full algebraic statement is given below:

Primal problem:

Max! $O_0 = 350 - 1.1X_1 - 1.25X_2 - 3X_3 - 7.5X_4 - X_5 - 2.5X_6 - .6X_7 - 3.0X_8$

Subject to $O_1 = 2000 - 12.5X_1 - 7.5X_2 - 6.0X_3 - 7.0X_4 - 15.0X_5 - 5.0X_6 - 4.0X_7 - 11.0X_8$

$O_2 = -50 - X_1 - X_2 - .5X_3 - .2X_4$

$O_3 = -50 - .25X_2 - X_3 - X_4$

$O_4 = -25 - X_5 - X_6 - .8X_7$

$O_5 = -25 - .2X_5 - .5X_6 - X_7 - X_8$

and $O_i \geq 0 , i = 1,.., 5$

$X_j \geq 0 , j = 1,.., 8$

Dual problem

Min! $J_0 = 350 - 2000 Y_1 - 50 Y_2 - 50 Y_3 - 25 Y_4 - 25 Y_5$

Subject to $J_1 = - 1.1 - 12.5 Y_1 + Y_2$

$J_2 = - 1.25 - 7.5 Y_1 - Y_2 - .25Y_3$

$J_3 = .3 - 6.0 Y_1 - .5Y_2 - Y_3$

$J_4 = - 2.5 - 7.0 Y_1 - .2Y_2 - Y_3$

$J_5 = - 1.0 - 15.0 Y_1 - Y_4 - .2Y_5$

$J_6 = - 2.5 - 5.0 Y_1 - Y_4 - .5Y_5$

$J_7 = - .6 - 4.0 Y_1 - .8Y_4 - Y_5$

$J_8 = - 3.0 - 11.0 Y_1 - Y_5$

and $Y_i \geq 0, i = 1,.., 5$

$J_j \leq 0, j = 1,.., 8$

The sectoral sub-problems, following the Dantzig-Wolfe decomposition method, are:

Sector 1

Max! $Z_1 = p_K \; (-1.1X_1 - 1.25X_2 - .3X_3 - 2.5X_4) - p_L \; (-12.5X_1 - 7.5X_2 - 6.0X_3 - 7.0X_4)$

Subject to $\sigma_2 = -50 - X_1 + X_2 - .5X_3 - .2X_4$

$\sigma_3 = -50 - .25X_2 + X_3 \quad X_4$

and $\sigma_i \geq 0, \quad i = 2,3$

$X_j \geq 0, \quad j = 1,..,4$

Sector 2

Max! $Z_2 = p_K \; (-1.0X_5 - 2.5X_6 - .6X_7 - 3.0X_8) - p_L \; (-15.0X_5 - 5.0X_6 - 4.0X_7 - 11.0X_8)$

Subject to $\sigma_4 = -25 - X_5 + X_6 - .8X_7$

$\sigma_5 = -25 - .2X_5 - .5X_6 + X_7 + X_8$

and $\sigma_i \geq 0, \quad i = 4,5$

$X_j \geq 0, \quad j = 5,..,8$

In the above expressions p_K and p_L are constants; in particular $p_K = 1$. Depending on p_L/p_K different optima to the sectoral sub-problems will be attained 1/. Designate the total capital and labour requirements of any of these optima by $-K_{ts}, -L_{ts}$ where t is the index of a given optimum, and s is the index of the sector (1 or 2).

1/ The optima may be extreme-point (vertex) or homogeneous solutions (see Dantzig and Wolfe, 1961). Homogeneous solutions indicate that the maximand of the sub-problem may be expanded without limit; in other words, the specific sectoral resource constraints do not preclude such an expansion. If such a situation occurred in the full problem, it would indicate that the problem was unbounded; but the solutions to the sectoral sub-problems are also subject to the constraints on the connecting resources, and thus homogeneous solutions are permissible. None such occur in the present problem.

<u>The master problem</u> can now be stated as follows (in Tucker's condensed format):

		J_0	J_A	J_B	J_D	J_E	J_F	J_H		
Min!		=	=	=	=	=	=	=		
Max!	$-\sigma_0 =$	350	$-K_A$	$-K_B$	$-K_D$	$-K_E$	$-K_F$	$-K_H$	$* \; p_K$	$(= 1)(= Y_0)$
	$0 \leq \sigma_1 =$	2000	$-L_A$	$-L_B$	$-L_D$	$-L_E$	$-L_F$	$-L_H$	$* \; p_L$	$(= Y_1)$
	$0 = \sigma_6 =$	-1	1	1	1				$* \; p_1$	
	$0 = \sigma_7 =$	-							$* \; p_2$	
		$(= \lambda^*)$	λ_A^*	λ_B^*	λ_D^*	λ_E^*	λ_F^*	λ_H^*		

(The interpretation in terms of ordinary algebraic expressions follows the interpretation of the full problem.)

In the above formulation, all the efficient vertex solutions are included in the master problem. If all these were in fact present, the solution to the master problem would at once yield the overall optimum. The algorithm is based, however, on just a partial list of such vertices which initially define but a single feasible starting solution. At any stage of the algorithm the optimum to the master problem yields a set of shadow prices; at these prices, all vertices with positive ___ weights have zero profits, while other vertices have negative profits; no positive profits can occur at such an optimum.

In order to test whether the current optimum to the master problem is also an overall optimum, it is attempted to find a new vertex that will show a positive profit at current shadow prices. Since p_1 and p_2 are given, a profitable new vertex ____ must have the highest possible algebraic value for the expression

$$(-p_K \cdot K_\lambda - p_L \cdot L_\lambda)$$

where p_K and p_L are also given The sectoral sub-problems select the vertex which maximizes the above expression in each sector. If the algebraic sum of p_1 and this maximum is <u>positive</u> for a sector, vertex λ is profitable and the current optimum to the master problem is not an overall optimum.

The new vertex is then included in the list of known vertices, and the optimization for the master problem is repeated. In the contrary case the overall optimum has been attained.

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