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THEORETICAL INTRODUCTION TO THE FINITE ELEMENT METHOD 1/

by

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Contents

Introduction

This theoretical introduction was meant to be an intensive course covering a scant two days and is therefore lacking in completeness. My intention was mainly to explain all those things which every FEM user should have heard at least once in order to be able to cope with certain contingencies which might arise in application.

This present version is a pilot version and therefore open to suggestions of improvement which the author will be happy to consider in order to ensure continually improving quality of this manual.

1 MATRIX METHODS

A matrix is defined as being a rectangular array of figures or symbols arranged in lines and columns. This configuration is turned into a matrix by the addition of square brackets. Assuming a matrix to have m lines and n columns it is shown as follows:

$$\underset{(m \times n)}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & \cdot & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \vdots \\ a_{m1} & \cdot & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Let me underline the fact that the number of lines (m) is always named first. Therefore, A is a (mxn) matrix.

In the following chapters mention will often be made of line or column matrices or vectors. Assuming m = 1 we have a <u>line matrix or a line vector</u>.

$$A = \lfloor a_{11} \quad a_{12} \cdots \quad a_{1j} \cdots a_{1n} \rfloor$$

If, however, we assume n = 1 we have a <u>column matrix or a column vector</u>.

$$A = \{a_{11} \quad a_{21} \cdots a_{j1} \cdots a_{m1}\} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{j1} \\ \vdots \\ a_{m1} \end{bmatrix}$$

There are some special matrices which I should like to mention briefly at this juncture.

Diagonal Matrix

$$\underset{(4/4)}{A} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ & a_{22} & 0 & 0 \\ & & a_{33} & 0 \\ \text{symmetrical} & & & a_{44} \end{bmatrix}$$

$a_{ij} = 0$ provided that $i \neq j$ and a_{ii} is not 0 in each case

An alternative notation would be

$$A = \lceil a_{11} \; a_{22} \; a_{33} \; a_{44} \rfloor$$

Identity Matrix

This matrix is a special case of diagonal matrix defined above. In the case of a 3 x 3 matrix, for instance, we have

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ \text{sym} & & 1 \end{bmatrix} = \lceil 1 \quad 1 \quad 1 \rfloor$$

Band Matrix

Whenever all entries (or elements) of a matrix which are not equal to zero are arranged around the main diagonal the designation 'band matrix' applies. For instance:

$$\underset{(n \times n)}{A} = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & a_{n,n-1} & a_{n,n} \end{vmatrix}$$

Triangular Matrix

A matrix is called either an upper (U) or a lower (L) triangular matrix if all its elements situated either above or below the main diagonal are equal to zero.

$$\underset{(n\times n)}{L} = \begin{bmatrix} a_{11} & 0 & 0 & \cdot & \cdot & 0 \\ a_{21} & a_{22} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & & \cdot & \cdot & a_{nn} \end{bmatrix}$$

Symmetrical Matrix

In a symmetrical matrix a_{ij} is always equal to a_{ji}. In linear structural mechanics, for instance, all stiffness matrices are symmetric.

Transposed Matrix

A transposed matrix is produced by exchanging lines for columns, as for instance

$$\underset{(2\times 3)}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Thus, a transposed matrix is

$$\underset{(3\times 2)}{A^t} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Moreover,

$$(A^t)^t = A$$

and, in the case of symmetric matrices,

$$A^t = A$$

Hyper- or Supermatrices

Larger matrices of, for instance, the size of 5,000 x 5,000 containing 25,000 entries necessarily have to be subdivided into smaller matrices, such as

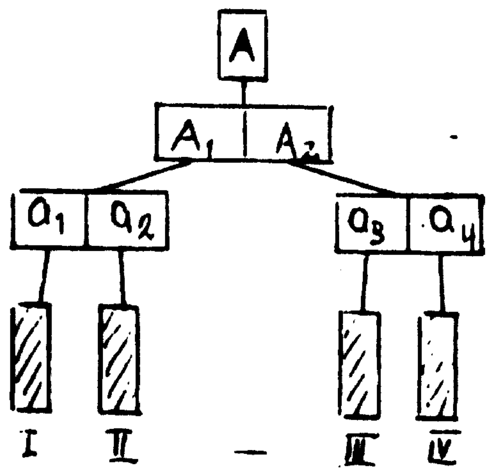
$$\underset{(3\times3)}{A} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

developing into

$$\underset{(2\times2)}{A_{11}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underset{(2\times1)}{A_{12}} = \{ a_{13} \quad a_{23} \}$$

$$\underset{(1\times2)}{A_{21}} = [\, a_{31} \quad a_{32} \,] \qquad \underset{(1\times1)}{A_{22}} = [\, a_{33} \,]$$

This subdivision into submatrices can, of course, be done in several stages. ASKA, for one, provides 3 stages.



Label

Matrix of adresses

Matrices of same

Numerical matrices

I through IV

If we apply this examplo to the 3 x 3 hypermatrix $\mathbf{A}$ mentioned above, matrix III would be our $A_{21} = [a_{31} \;\; a_{32}]$

1.2 Calculations Involving Matrices

In calculation it is possible to treat matrices just like you usually treat numerical data. In the following, we give the definitions required for our purpose.

Equality

$$\mathbf{A} = \mathbf{B}$$

means that for all i and j $a_{ij} = b_{ij}$.

Addition and Subtraction

If

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

then

$c_{ij} = a_{ij} + b_{ij}$

In the case of subtraction, consequently, we have

$c_{ij} = a_{ij} - b_{ij}$

Matrix Multiplication

If a matrix is to be multiplied by a factor c every single entry must be multiplied by c, e.g.

$$c\mathbf{A} = [ca_{ij}]$$

In multiplying two matrices it is a conditio sine qua non that their dimensions be compatible. If, for instance, (mxn)matrix $\mathbf{A}$ is to be multiplied by (oxp) matrix $\mathbf{B}$ it is requisite that n = o, i.e. the number of lines n contained in $\mathbf{A}$ must be equal to the number of columns o contained in $\mathbf{B}$.

Thus,

$$\underset{(m\times n)}{\mathbf{A}}\;\underset{(o\times p)}{\mathbf{B}} = \underset{(m\times p)}{\mathbf{C}}$$

and

$$c_{ij} = \sum_{r=1}^{o} a_{ir} b_{rj} \qquad i = 1,2 \cdots m;\; j = 1,2 \cdots p$$

A simple example would be

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{11} & a_{13} \end{bmatrix} \{ b_{11} \quad b_{21} \quad b_{31} \}$$

$$= \{ a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31} \qquad a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31} \}$$

If all single entries (e.g. a_{22}, b_{31} and $a_{22}b_{31}$) are regarded as sub-matrices this example is also applicable to sub-matrices.

Solution of Linear Matrix Equations in Static Problems

For reasons of economy linear matrix equations occurring in structural mechanics are solved by solving the coefficient matrix (e.g. the stiffness matrix) instead of by a genuine inversion of the matrix concerned, as the former way necessitates the least number of numerical operations. There are various matrix reduction techniques whose applicability has been established. In ASKA, Cholesky's technique is used on symmetric matrices (cf. e.g. R. Zurmühl, "Matrizen und ihre technischen Anwendungen", Springer-Verlag). Cf. also figs 1.1, 1.2, 1.3 and 1.4.

The Concept of ASKA

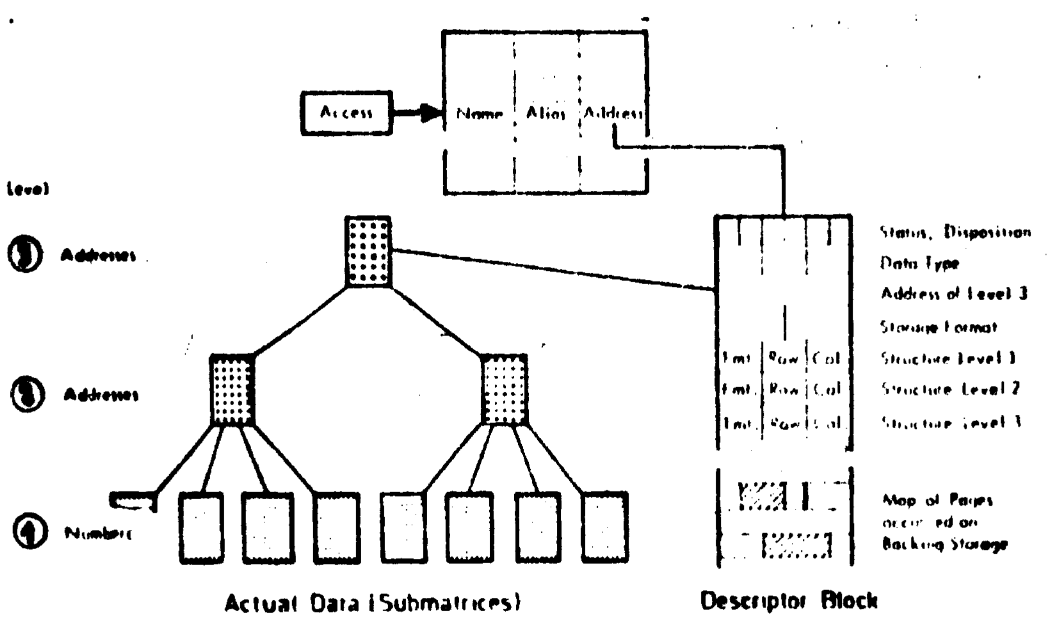


Fig. 1.1

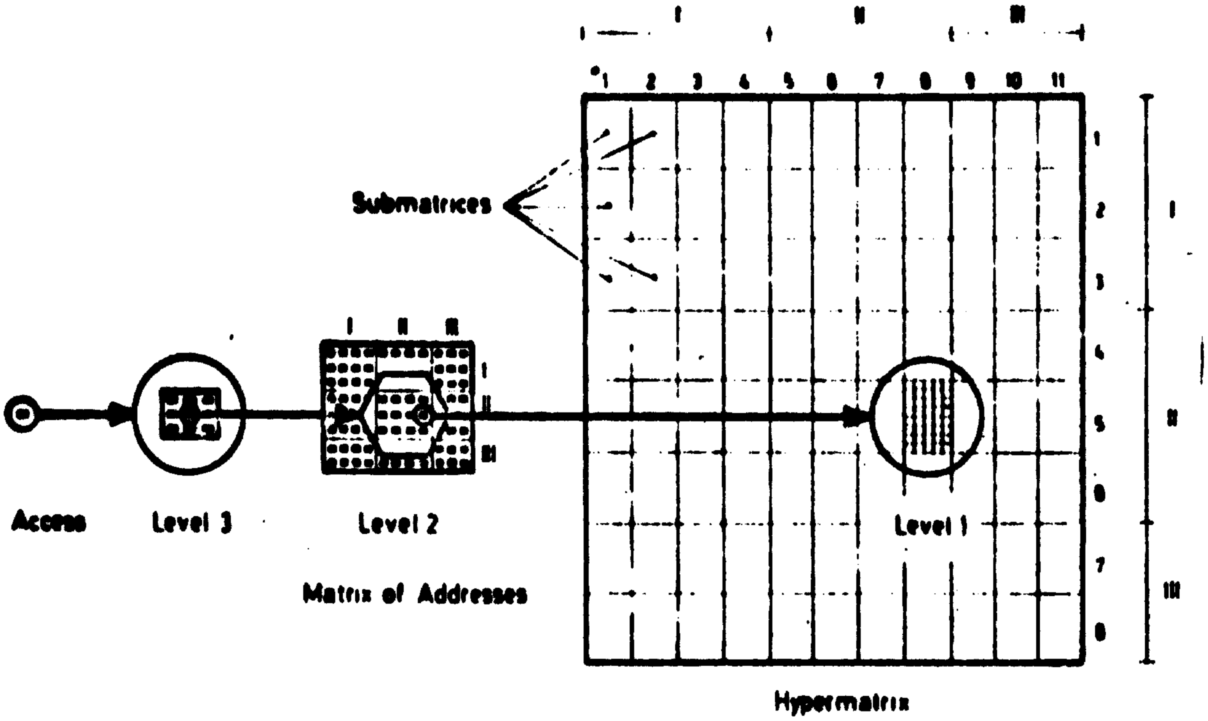
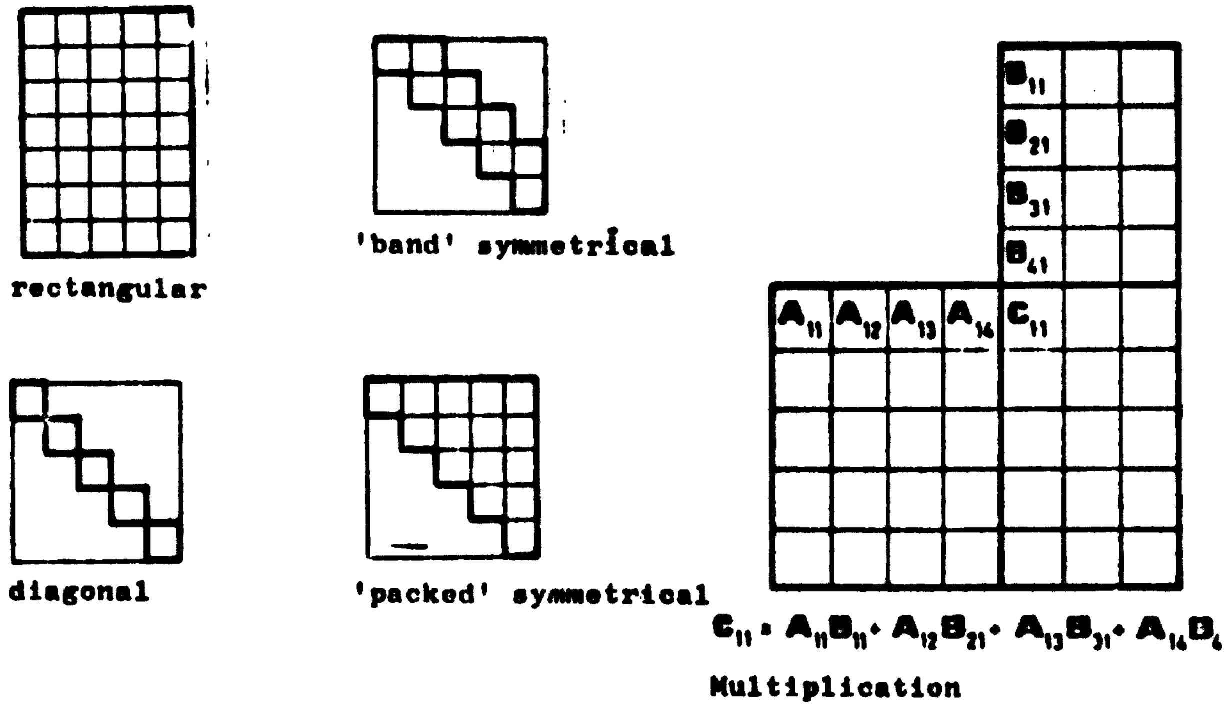


Fig. 1.2 Recursive Sub-Matrix Technique



Storage formats

Fig. 1.3 Hypermatrices

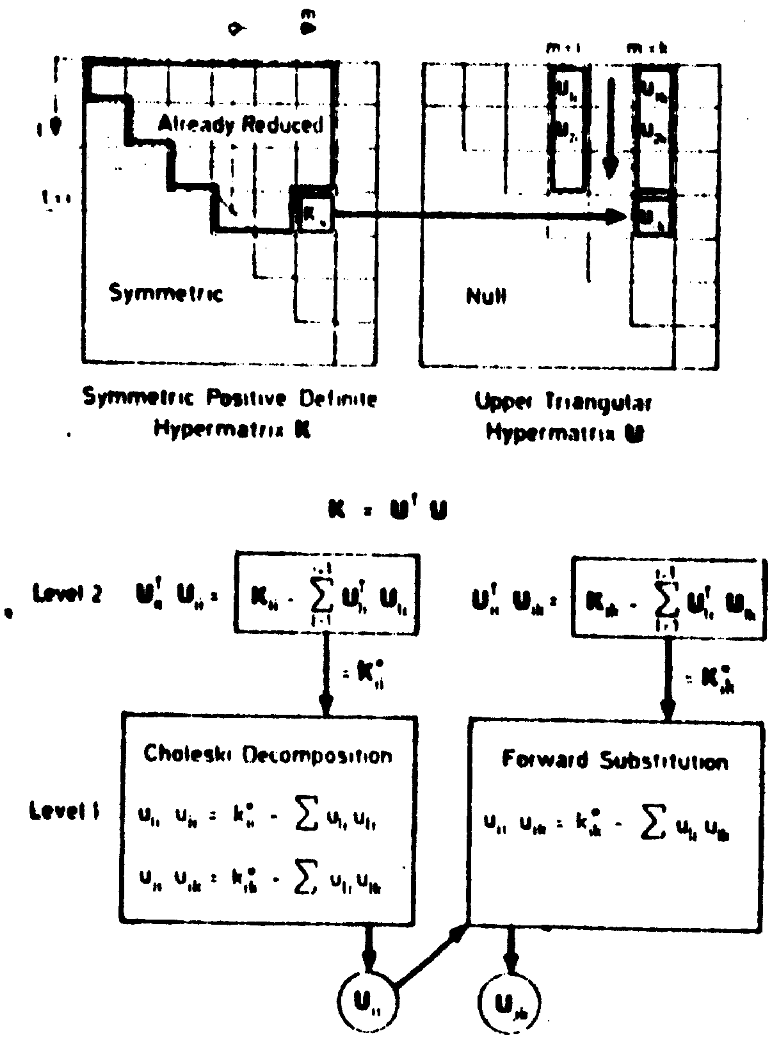


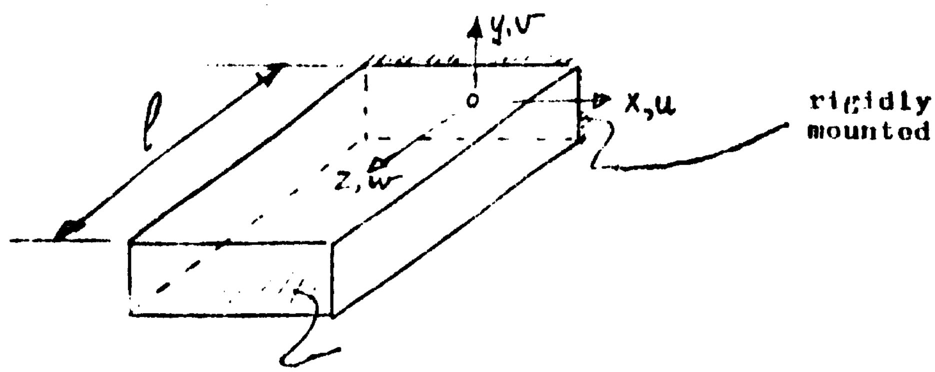
Fig. 1.4 Recursive Choleski Factorization

2. ELASTICITY EQUATIONS IN STATICS

There are three basic conditions used in static problems to establish the requisite systems of equations:

- Equilibrium or static consistency;
- Kinematic consistency;
- Stress-strain relations.

Static and/or kinematic conditions occur among boundary conditions as well. One possible occurrence of static and kinematic boundary conditions is best illustrated by means of a hollow box.



rigid end plate

At $z = 0$ the hollow box is rigidly mounted. The web is infinitely stiff towards deformations in its plane but allows deformations perpendicular to its-plane. Here are the boundary conditions:

$z = 0$

$u = v = w = 0$, i.e. kinematic conditions only.

$z = l$

$\sigma_{zz} = 0$, i.e. static condition.

$\left.\begin{array}{ll} \partial v/\partial y = 0, & \text{applying to vertical walls.} \\ \partial u/\partial x = 0, & \text{applying to horizontal walls.} \end{array}\right\}$ Kinematic condition then

$$\varepsilon = D u \qquad (2.6)$$

(this really being applicable only to minor displacements with D being defined as in equation (2.2)).

This demonstrates that it is necessary to know the displacement vector u in every point in order to describe completely the state of displacement in a general 3-D continuum.

Kinematic consistency can be defined verbally as follows:

> A state of displacement is to be deemed kinematically consistent whenever neighbouring parts neither diverge nor penetrate one another after deformation.

2.3 Stress - Strain Relations

For linear-elastic materials these relations can be expressed as follows:

$$\sigma = E \varepsilon \qquad (2.7)$$

with E representing a 6 x 6 elasticity matrix reflecting the material properties of the material concerned; anisotropic behaviour is not ruled out. This matrix is positively definite. (Incompressible materials constitute an exception.)

Applying Hooke's Law, we have

$$E = \frac{E}{(1+\nu)(1-2\nu)} \begin{vmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ \text{Symmetrisch} & & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{vmatrix} \qquad (2.8)$$

E being Young's modulus and ν being Poisson's Constant.

For isotropic materials in which heat expansion is e ual in all directions equation (2.7) can be expanded as follows:

$$\sigma = E\varepsilon + \alpha T\, E_T - E\varepsilon_I \qquad (2.9)$$

with α constituting the heat transfer coefficient, T representing temperature alteration, ε_I being the 6 x 1 column vector of initial strains and

$$E_T = \frac{E}{1-2\nu}\left\{-1 \quad -1 \quad -1 \quad 0 \quad 0 \quad 0\right\} \qquad (2.10)$$

Equation (2.9) makes it possible to arrive at total strains ε_t by multiplying both the right and the left side by E^{-1} (i.e. the inverted matrix of E, $E^{-1}E = I_6$).

$$\varepsilon_t = E^{-1}\sigma - \alpha T\, E^{-1}E_T + \varepsilon_I \qquad (2.11)$$

or

$$\varepsilon_t = \varepsilon_E + \varepsilon_T + \varepsilon_I \qquad (2.12)$$

with

$$\varepsilon_E = E^{-1}\sigma \qquad \text{(elastic strains)} \qquad (2.13)$$

$$\varepsilon_T = \cdot\alpha T\left\{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\right\} \qquad (2.14)$$

$$\varepsilon_I = \left\{\varepsilon_{Ixx} \quad \varepsilon_{Iyy} \quad \varepsilon_{Izz} \quad \varepsilon_{Ixy} \quad \varepsilon_{Iyz} \quad \varepsilon_{Izx}\right\}. \qquad (2.15)$$

$$\mathbf{E}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix} \qquad (2.16)$$

2.1 Static Consistency

In the case of a three-dimensional continuum the equilibrium or static-consistency conditions can be defined as follows:

$$\mathbf{D}^t \sigma + \boldsymbol{\omega} = 0 \qquad (2.1)$$

with representing the 3 x 6 matrix of differential operators,

$$\mathbf{D}^t = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \qquad (2.2)$$

and representing the 3 x 1 column vector of volume forces.

$$\boldsymbol{\omega} = \{ \omega_x \quad \omega_y \quad \omega_z \} \qquad (2.3)$$

We can state generally that

> all forces acting internally and externally must be balanced.

2.2 Kinematic Consistency

This type of consistency is geometric in nature. If we define all deformations of a continuum caused by external loads or temperature gradients by means of the displacement vector

$$\mathbf{u} = \{ u_x \quad u_y \quad u_z \} \qquad (2.4)$$

and the appropriate strain vector

$$\boldsymbol{\varepsilon} = \{ \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{zx} \} \qquad (2.5)$$

If we now take into account the fact that external work must always be balanced by interna. work, we have

$$\delta W = \delta U \quad \text{und} \quad \delta W^* = \delta U^*$$

Therefore,

$$\int_V \mathbf{w}^t \delta \mathbf{u}\, dV + \int_F \boldsymbol{\phi}^t \delta \mathbf{u}_s\, dF = \int_V \boldsymbol{\sigma}^t \delta \boldsymbol{\varepsilon}_L\, dV \qquad (3.1)$$

$$\int_V \mathbf{u}^t \delta \mathbf{w}\, dV + \int_F \mathbf{u}^t \delta \boldsymbol{\phi}\, dF = \int_V \boldsymbol{\varepsilon}_L^t \delta \boldsymbol{\sigma}\, dV \qquad (3.2)$$

3.1 Virtual Work

In the course of the explanations of the energy theorems given above we have assumed that displacements occurred because of real loads. This limitation, however, is not necessary. The equations given above demonstrate that δW and δU can be expressed independently of δP and δT. Therefore, it is possible to select an expression of virtual work, such as

$$\mathbf{u} + \delta \mathbf{u}$$

the only condition being that $\delta \mathbf{u}$ is a kinematically consistent deformation.

An elastic body is considered to be in equilibrium under any given load, including temperature loads, if

$$\delta W = \delta U \qquad (3.3)$$

can be applied to any individual virtual displacement $\delta \mathbf{u}$ of a kinemcticaliy consistent state of displacement. This equation represents the principle of virtual work. It can be defined as follows:

> External virtual work caused by external loads and involving virtual displacements is equal to internal virtual work performed by stress and involving virtual strains provided that the stresses are statically consistent with the outer loads.

3. ENERGY THEOREMS

In 1954, Professor J. H. Argyris stated in his book, 'Energy Theorems and Structural Analysis' that all energy theorems can be reduced to two basic principles:

- The principle of virtual work (or of virtual displacement); and
- The principle of complementary virtual work (or of virtual forces).

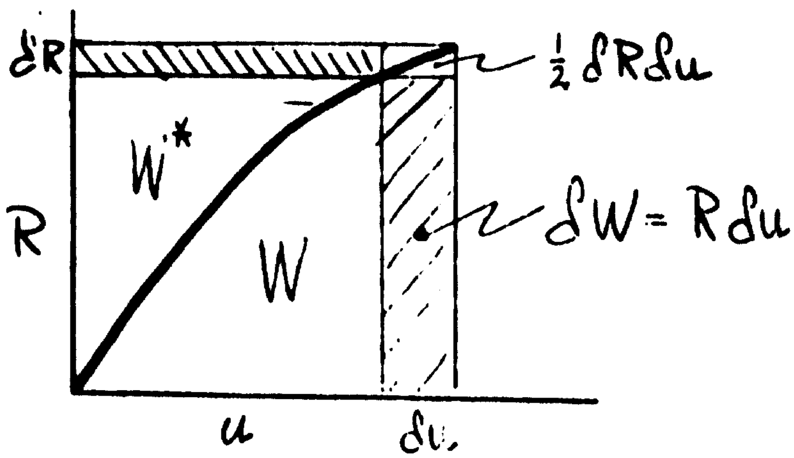
These two principles constitute the basis of the strain energy method employed in structural mechanics.

To begin with, we shall deal merely with minor displacements or strains, with all equations occurring representing linear behaviour (i.e. all equations can be added together). This does not mean, however, that we can deal with stress-strain relations only, to the exclusion of all others; we presuppose, however, that all relations change only monotonously. This makes clear that, although strains and displacements may be superimposed this is not always possible with stresses. Moreover, there is a specific approach to each problem.

To begin with, let us deal with a 3-D body under the following loads:

Balancing volume forces;	$\mathbf{w}$	(per unit volume)
Surface forces;	$\mathbf{\Phi}$	(per unit area)
Singular forces.	$\mathbf{R}$	

Looking at the displacement diagram



the change in work δW can now be expressed as

$$\delta W = R\delta u + \frac{1}{2}\delta R\delta u$$

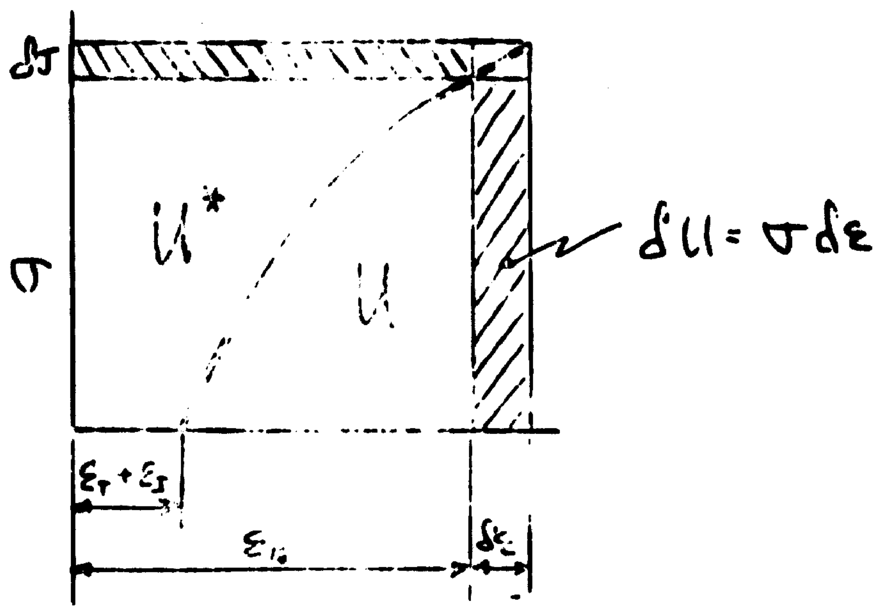
or, if higher-order members are left out, as

$$\delta W = R\delta u = \int_V w^t \delta u \, dV + \int_F \Phi^t \delta u \, dF$$

In complementary work, of course, the following applies:

$$\delta W^* = u\delta R = \int_V u^t \delta w \, dV + \int_F u^t \delta \Phi \, dF$$

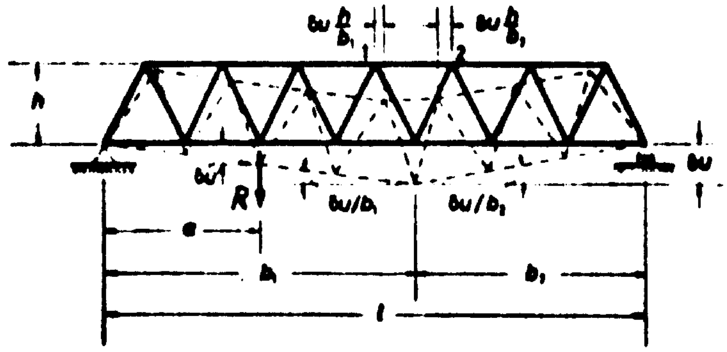
We can establish similar equations to express the strain energy (U and U*):



$$\delta U = \int_V \sigma^t \delta \varepsilon_G \, dV$$

$$\delta U^* = \int_V \varepsilon_G^t \delta \sigma \, dV$$

Let us apply this principle to a little example in order to confirm our ideas.



Our example is a statically determined latticework structure in which the force exerted on the rod 1.2 is to be determined. Our selected virtual displacement δu permits only of a kinematically consistent shortening of rod 1.2. All other rods are not subjected to any alteration of their respective load states.

Vertical displacement under load R is: $\delta u \frac{a}{b_1}$

Shortening of the rod 1.2 is: Δl_{12}

$$\Delta l_{12} = -\delta u \left(\frac{h}{b_1} + \frac{h}{b_2} \right) = -\delta u \frac{lh}{b_1 b_2}$$

The rod force to be determined is: N_{12}

We now apply the principle of virtual work by

$$R \delta u \frac{a}{b_1} = - N_{12} \delta u \frac{lh}{b_1 b_2}$$

or

$$\underline{\underline{N_{12} = - \frac{R a b_2}{l h}}}$$

3.2 The Unit Load Method

A special version of the principle motioned above is called the unit load method. Later on, this method is used very frequently in compiling element matrices.

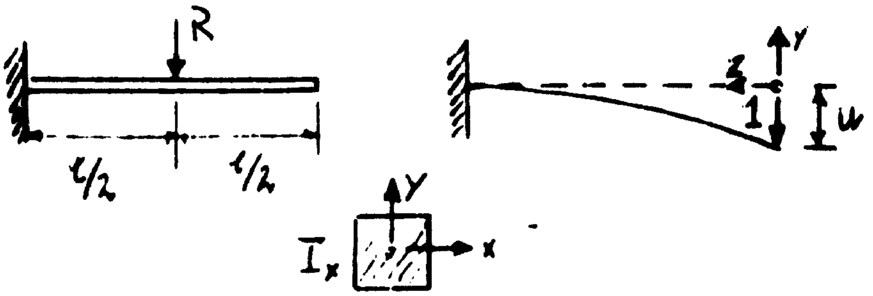
Instead of a random load, a singular force is now applied in the direction of the displacement to be determined:

$$1\,\delta u = \int_V \sigma \varepsilon \, dV \qquad (3.4)$$

In this equation,

δu = displacement in the direction of the load;
ε = true (or real) strain vector;
σ = stress vector, which is merely required to be in balance with the load.

The following exemplifies the application of the unit load method:



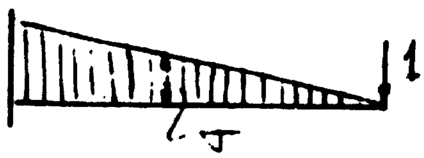
The displacement u occurring at the tip of the beam is to be determined.

The true strains are:

$$\varepsilon_{zz} = \frac{R}{EI_x}\left(z - \frac{l}{2}\right)y \qquad \text{für } \frac{l}{2} < z < l$$

$$\varepsilon_{zz} = 0 \qquad \text{für } 0 < z < \frac{l}{2}$$

The stress vector:



$$\sigma_{zz} = \frac{1 \cdot z}{I_x}\,y$$

Therefore, the displacement can be determined by

$$1 \cdot u = \int_V \sigma \varepsilon \, dV = \int_F \left[\int_{l/2}^{l} \frac{R}{EI_x}\left(z - \frac{l}{2}\right) z \, dz \right] \frac{1}{I_x} y^2 \, dF$$

$$\underline{1 \cdot u} = \frac{R}{EI_x}\left[\frac{z^3}{3} - \frac{l z^2}{4}\right]_{l/2}^{l} = \underline{\frac{5}{48}\,\frac{R l^3}{EI_x}}$$

4. IDEALIZATION - STRUCTURAL MODEL AND ELEMENTS

The most important step of the matrix method in structural mechanics is to determine the physical or mathematical model of the structure. If the displacement method is used, this model will fulfil everywhere the condition of kinematic consistency but will in most cases be statically consistent only in the nodes. (Conditions are reversed if the dual method - the force method - is used.)

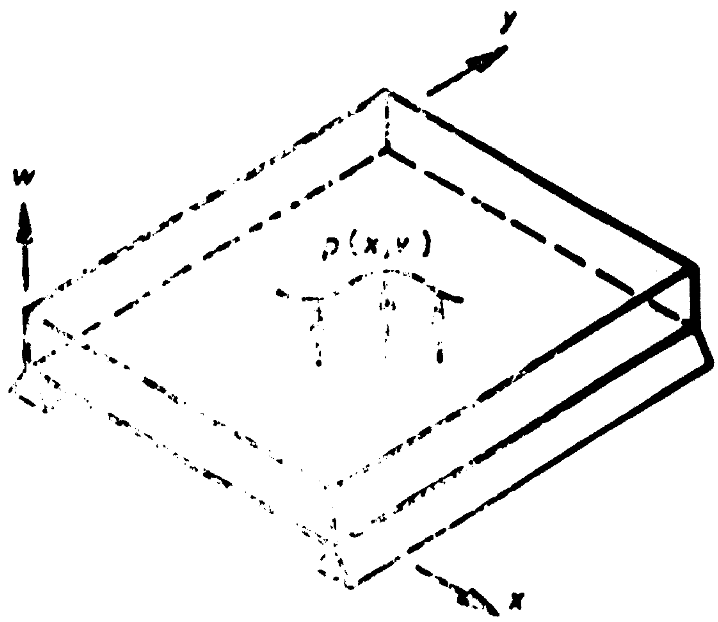
The model is devised by subdividing the structure into a finite number of finite elements. Kinematic consistency is a prerequisite for each element, whereas static consistency generally is required only at the nodal points of each element. In some structures, e.g. simple framework structures, defining a physical model is very easy, whereas three-dimensional structures in most cases require some compromise solutions. Subdividing a structure into finite elements is somewhat difficult. Only a constant user of this method will be able to circumvent these difficulties economically, successfully, and in accordance with engineering requirements. Beginners are mostly inclined to work far too hastily. Always keep in mind the fact that it is not the structure itself which is computed but merely a model selected for the purpose. Mistakes committed in devising the model, such as faulty boundary conditions, may produce a completely erroneous result. Should the error be found out later you will have spent a lot of money without being able to show something for it. Should the error remain undiscovered, which is perfectly possible if the engineers concerned are inexperienced, malfunctions in the finished structure may cause a lot of expense. It is, therefore, our opinion that there should be no unnecessary compromises when devising the model. The determination to save DM 1,000.- in man-hours may cost ten times as much in alterations which may be found necessary later on.

Assuming that the appropriate model has been defined we now enter the second difficult phase, that of checking the results. Here, too, we meet a number of unexpected surprises which in most cases can be overcome through experience.

Before going any further it is probably best to explain briefly the major characteristics of the Finite Element Method (FEM). The following figures show a diaphragm being calculated according to the classical method (i.e., the finite difference method) and according to FEM.

CLASSICAL THEORY

1 Mathematical Model

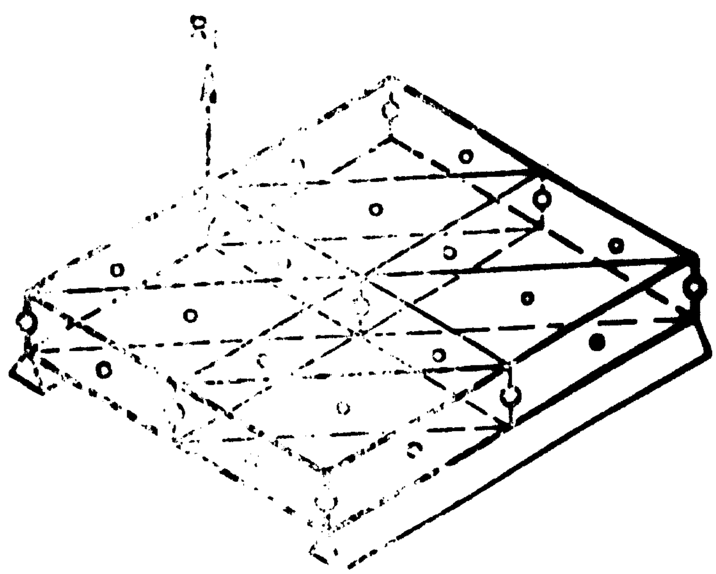


2 Problem Definition

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(x,y)}{D}$$

FINITE ELEMENT THEORY

1 Idealized Model



2 Problem Definition

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

The figures show that with the classical method the diaphragm's deformation is expressed by the well-known biharmonic equation, which in turn is a function of the two coordinates x and y. Moreover, the deformation w (x,y) is dependent on the load imposed p (x,y). D defines the material and geometric properties of the diaphragm. Now, if you want to define the same diaphragm by means of a FEM model it is first described by a finite number of nodes located in the centre of the diaphragm plane. Those nodes (often called a mesh) are then connected by a corresponding number of bonding elements, thus constituting a complete model of the diaphragm. The external load defined by p(x,y) in the classical method can be incorporated into the FEM model only as a nodal load (R_i). These nodal loads must be kinematically consistent with the actual surface load. The load is to be deemed kinematically consistent if the work performed as nodal point displacement by the nodal point loads, i.e. $\sum R_i r_i$, is equal to the work performed as corresponding internal displacements in the element itself by the distributed surface loads. An example to explain this point will be given in chapter 7.

Once the external load is defined kinematically consistent

$$\mathbf{R} = \{R_1 \; R_2 \; \cdots \; R_n\}$$

the stiffness matrices of each individual element $\mathbf{k}_i$ are compiled (cf. also chapter 5). The individual stiffness matrices are then used to compose the global stiffness matrix ($\mathbf{K}$) of the finite element structure. The linear system of equations pertaining to the finite-element model selected now reads as

$$\boxed{\mathbf{K}\underline{\mathbf{r}} = \mathbf{R}} \qquad (4.1)$$

With this equation established it should be clear once and for all that generating the global stiffness matrix $\mathbf{K}$ is the paramount factor in the selection of the correct physical model. As has been mentioned before, the global stiffness matrix is composed of the added entries regarding the individual elements (k_i). Consequently, the quality of the model selected (i.e. K) depends mainly on the type of elements (diaphragms, discs, beams) available to the modelling engineer. One might almost say that the larger the number of element types available to the engineer the better he will be able to adapt his model to the actual physical conditions. (Our ASKA system offers about 50 different types of element.) Engineers who use FEM only sporadically maintain that such a wide range of elements is only apt to cause confusion. This view, however, is caused by the fact that this particular type of user does not know enough about the facts and problems of correct FEM application. (For instance: IKOSS have solved problems requiring more than 10 different types of element to ensure a sufficiently accurate model.)

In conclusion I should like to stress again that

> beginners in the use of the Finite Element Method should never attempt in a sort of valiant effort to compute alone any structure which is actually to be erected later on.

Although experience gained by working on one's own is most valuable it is too costly in terms of money and time!

5 ELEMENT STIFFNESS

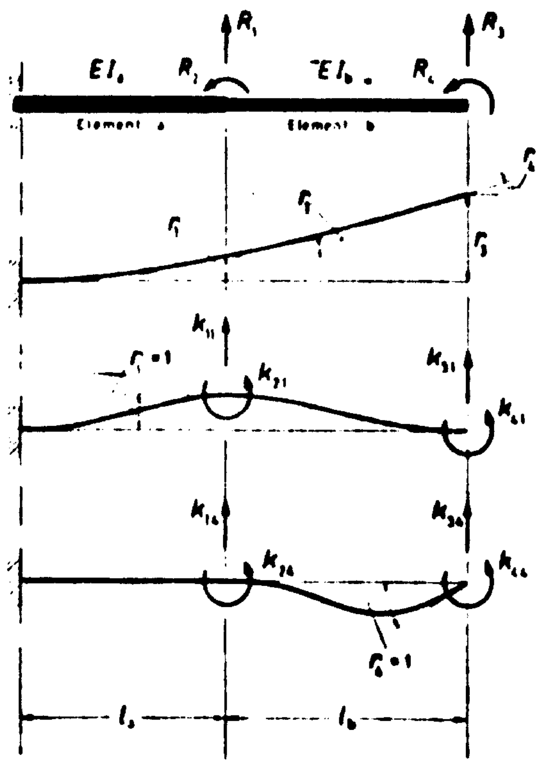
The significance of having good finite elements to be used as components when modelling a given structure in theory has already been underlined in the preceding chapter. In view of the fact that a multitude of element types has already been described in literature it only remains for us to mention the way in which an element stiffness is generated.

Generally speaking, there are several ways by which to arrive at an element stiffness matrix:

1) Unit load method or unit displacement method.
2) Castigliani's Theorem.
3) Solving relevant differential equations.
4) Inversion of flexibility matrix.

Of all the methods mentioned above method 1) is most generally employed (cf. also equation (3.4)) because it shows all stiffnesses (i.e. forces) to be determined in the direction of the predetermined unit displacements.

Let us consider an easy example (a cantilever beam) in order to improve our understanding of the physical significance of the term 'stiffness'.



The figure above shows that four degrees of freedom (i.e. the same number of unknowns used in the displacement method) are fully sufficient to account for all kinematic possibilities.

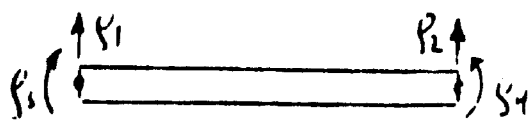
$$\mathbf{r} = \{ r_1 \quad r_2 \quad r_3 \quad r_4 \} \tag{5.1}$$

Moreover, both the stiffness matrices $\mathbf{k}_a$ and $\mathbf{k}_b$ (element a and b) as well as the global stiffness matrix $\mathbf{K}$ are symmetric matrices (proof by means of Betti's Theorem). Therefore,

$$k_{ij} = k_{ji} \quad \text{und} \quad K_{ij} = K_{ji} \tag{5.2}$$

Furthermore, the non-singular global stiffness matrix $\mathbf{K}$ must necessarily be a (4 x 4) type of matrix, i.e. its dimension must be equal to the number of unknowns.

Let us first consider element a individually.



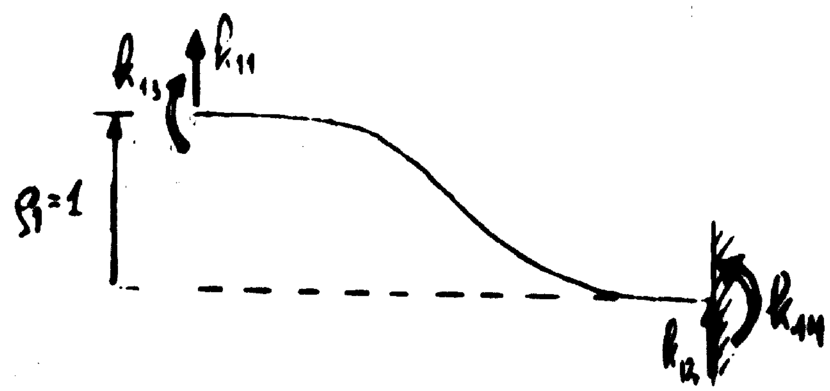
When considered in this manner, i.e. disregarding the existing boundary conditions, element a has four degrees of freedom. It would be possible, of course, to incorporate the axial force of the rod by way of an additional degree of freedom, but this is not relevant to our problem. To differentiate between the element freedoms and those on the global level ($\mathbf{r}$) we use the Greek letter 'Rho' in lower case.

$$\boldsymbol{\rho}_a = \{ \rho_1 \quad \rho_2 \quad \rho_3 \quad \rho_4 \}_a \tag{5.3}$$

In order to find the first line of our stiffness matrix $\mathbf{k}_a$, i.e. vector

$$\mathbf{k}_{1j} = \{ k_{11} \quad k_{12} \quad k_{13} \quad k_{14} \} \tag{5.4}$$

it is merely necessary to carry out a unit displacement in the direction of degree of freedom ρ_1 while immobilizing all other end or nodal points.



(Note: $\rho_2' = 0$.)

The physical significance of the individual k_{ij} is that they represent the actual forces required to generate the displacement $\rho_1 = 1$ only.

Now, the unit displacement method (cf. also equation (3.4) defining the analogous unit load method) can be used to arrive at the forces (i.e. stiffnesses) to be determined.

$$\boxed{1\, k_{ij} = \int_V \sigma_i\, \varepsilon_j\, dV} \qquad (5.5)$$

If we apply this procedure to k_{11} we have

$$1\, k_{11} = \int_V \sigma_1\, \varepsilon_1\, dV$$

σ_1 merely being required to be statically balanced.

$$v = 3\left(\frac{z}{l}\right)^2 - 2\left(\frac{z}{l}\right)^3$$

In accordance with the general beam theory,

$$\sigma = \frac{M}{I} y = E v'' y$$

$$\varepsilon = v'' y$$

Moreover,

$$v'' = \frac{d^2 v}{dz^2} = \frac{12}{l^2}\left(\frac{1}{2} - \frac{z}{l}\right) \qquad (5.6)$$

and therefore

$$k_{11} = \int_F \left[\int_0^l E (v'')^2 dz \right] y^2 dF$$

Now we know that

$$I_x = \int_F y^2 \, dF$$

which gives us a sufficient basis to solve the following integral.

$$\underline{k_{11}} = EI \int^{l_1} (v'')^2 \, dz = \underline{12 \frac{EI}{l_1^3}}$$

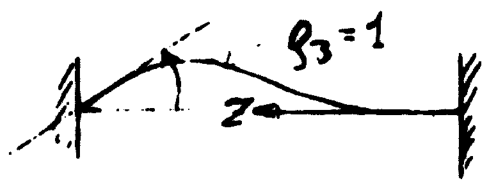
Furthermore,

$$k_{22} = k_{11}$$

It is easy to determine the force k_{33} ($= k_{44}$) in a similar way.

$$k_{33} = k_{44} = 4 \frac{EI}{l_1}$$

Now, in order to arrive at k_{13}, for instance, the mode of displacement of the beam must be defined in correspondence with $\rho_3 = 1$.



$$v_3 = -l \left(\frac{z}{l}\right)^2 \left[1 - \frac{z}{l}\right]$$

Counter-clockwise rotation is defined as negative.

Now, the integral reads

$$1\, k_{13} = \int_V \sigma_1 \varepsilon_3 \, dV = \int_V EI (v_1'')(v_3'') \, dz$$

As

$$v_3'' = \frac{d^2 v_3}{dz^2} = -\frac{2}{\ell}\left(1 - \frac{3}{\ell} z\right) \qquad (5.7)$$

Consequently,

$$\underline{k_{13}} = \underline{k_{31}} = -EI \frac{24}{\ell_a^3} \int^{\ell_a} \left(\frac{1}{2} - \frac{2}{\ell}\right)\left(1 - \frac{3}{\ell} z\right) dz = \underline{-6 \frac{EI}{\ell_a^2}}$$

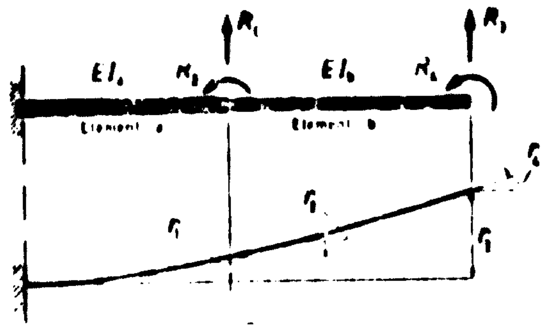
The entries still missing from element stiffness matrix k_a to be determined can be found by way of a similar procedure.

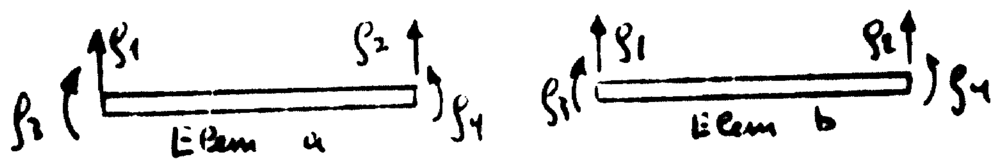
For completeness' sake we give you below the entire matrix pertaining to element a. To arrive at the matrix for element b you merely replace ℓ_a by ℓ_b and I_a by I_b.

$$k_a = \frac{EI_a}{\ell_a^3} \begin{bmatrix} 12 & -12 & -6\ell_a & 6\ell_a \\ & 12 & 6\ell_a & -6\ell_a \\ & & 4\ell_a^2 & -2\ell_a^2 \\ \text{Sym.} & & & 4\ell_a^2 \end{bmatrix} \qquad (5.8)$$

Please note that the matrix shown above does not take into account any shear deformations.

As all stiffness entries of the global matrix K can be generated by simple addition of the individual element stiffness matrix components (i.e. in k_a and k_b) the 4 x 4 matrix K can be compiled manually. To deepen the operator's insight this procedure can be illustrated by the following figure.





The figure shows that entry k_{44} corresponding to freedom r_4 is arrived at by adding up the two entries $(k_{22})_a$ and $(k_{44})_b$, which means

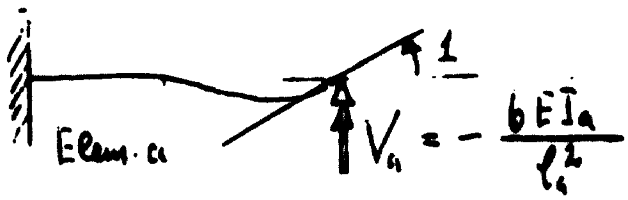
$$K_{11} = (k_{22})_a + (k_{11})_b = 12\left[\left(\frac{EI}{\ell^3}\right)_a + \left(\frac{EI}{\ell^3}\right)_b\right]$$

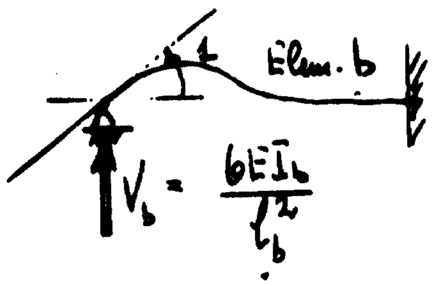
The remaining entries are arrived at in a similarly simple manner, for instance $K_{44} = (k_{44})_b$.

To convince those who are not completely sure about the correctness of this procedure we demonstrate below how, for example, the entry concerned can be calculated directly.

For this purpose, we assume $r_2 = 1$.

The resultant force K_{12} can be simply determined by superposition of supports.





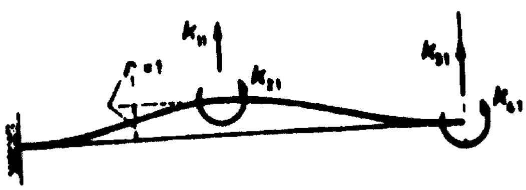
Therefore,

$$K_{12} = V_a + V_b = -6\left(\frac{EI}{l^2}\right)_a + 6\left(\frac{EI}{l^2}\right)_b$$

Considering element stiffnesses it is obvious that

$$V_{12} = (k_{54})_a + (k_{53})_b$$

which again gives us the same result.



6 SELECTING SUITABLE ELEMENTS

Given the large number of element types to be found in literature it is not always easy to decide what type of element to use (to program into) a FEM program. I shall not make an attempt to submit clear proposals in this matter - the opinions held by experts are far too divergent for that. I shall merely state a few important findings.

For each element a deformation mode is to be selected. In doing so, two requirements are of major importance from the theoretical point of view:

- Completeness of the deformation mode;
- Kinematic consistency of the element boundaries.

To ensure correct registration of all movements of a rigid body, moreover, it is requisite that the sum of all interpolation functions applying to each point of the element range be equal to one. If, for instance, the vector field u of the displacements within a two-dimensional element are expressed as follows

$$u(x,y) = \sum_{i=1}^{n} \varphi_i(x,y)\, \varrho_i \tag{6.1}$$

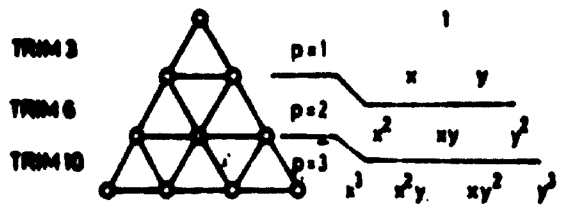
then, necessarily,

$$\sum_{i=1}^{n} \varphi_i(x,y) = 1 \tag{6.2}$$

Let us now, for simplicity's sake, merely consider polynomial deformation modes of Lagrange's type (i.e. having no derivations as freedoms) applied to triangular and rectangular elements which we assume to have membrane stiffnesses only.

Triangular membrane elements - called TRIM's in ASKA terminology - have a great advantage in that their displacement mode can be expressed as a complete polynomial. This means that the functional variation is independent of linear coordinate transformations.

If we designate the order of the displacement modes by p the following figure gives a good survey.



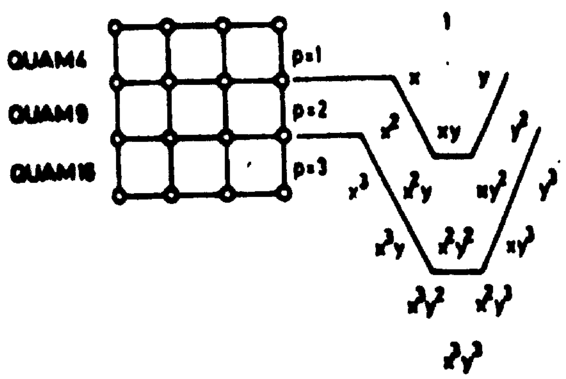
The figure shows that, for instance, in the case of TRIM 3 (i.e. three nodes) the vector entries of $u\ (x, y)$ are

$$u_x = \varphi_1 x + \varphi_2 y + \varphi_3$$

$$u_y = \varphi_4 x + \varphi_5 y + \varphi_6$$

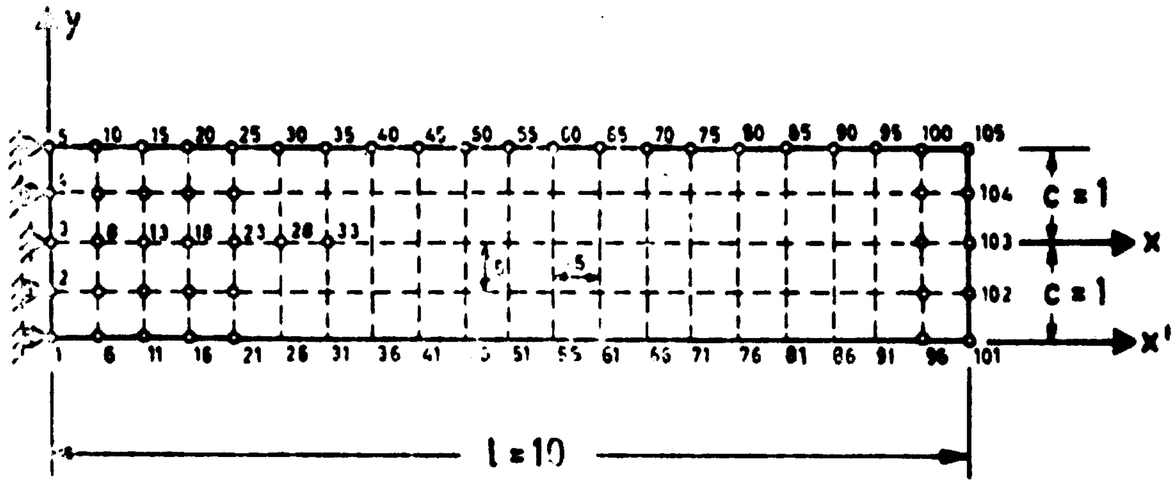
These six parameters φ_i are arrived at by using the given six possible nodal point displacemente to set up six equations which then can be solved in a more or less simple manner.

Representing quadrangular elements - QUAM in ASKA parlance - in the same manner as above, the figure is



On the one hand, the members of the p^{th} order of this displacement mode are complete, but they also contain polynomial members up to the $2p^{th}$ order. This means losing the invariant rotation property.

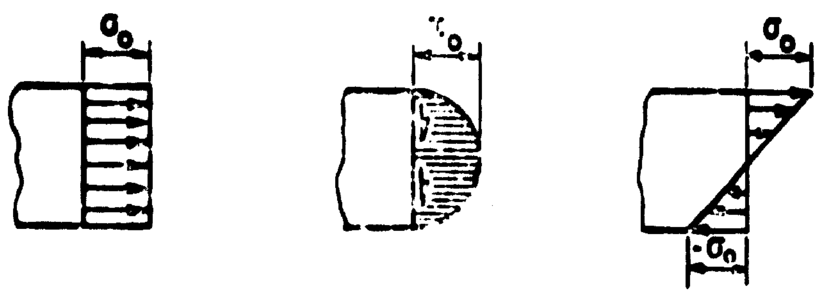
To demonstrate the advantages of higher-order elements we give the example of a cantilever beam below.



$$t = .5$$

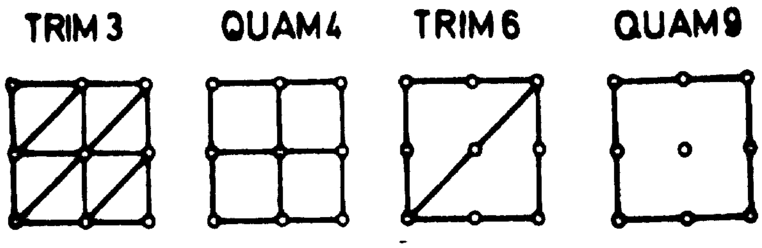
$$A = t \cdot 2 \cdot c = .5 \times 2 \times 1 = 1.$$

$$I_{zz} = \frac{1}{12} t (2c)^3 = \frac{1}{3}$$



Load case 1 Load case 2 Load case 3

The idealizations and element types selected are



All idealizations contain the same number of unknowns - 164. Moreover, it is simple to determine that the analytical values of the three load cases at the free end of the beam are as follows:

<u>Load case 1</u>

$$\sigma_{xx} = \sigma_0 = 12$$
$$\sigma_{yy} = \tau_{xy} = 0$$

<u>Load case :</u>

$$\sigma_{xx} = \sigma_{yy} = 0$$
$$\tau_{xy} = -4.5$$

<u>Load case 3</u>

$$\left.\begin{aligned} \sigma_{xx} &= 12\,\frac{y}{c} \\ \sigma_{yy} &= \tau_{xy} = 0 \end{aligned}\right\} \quad \text{everywhere}$$

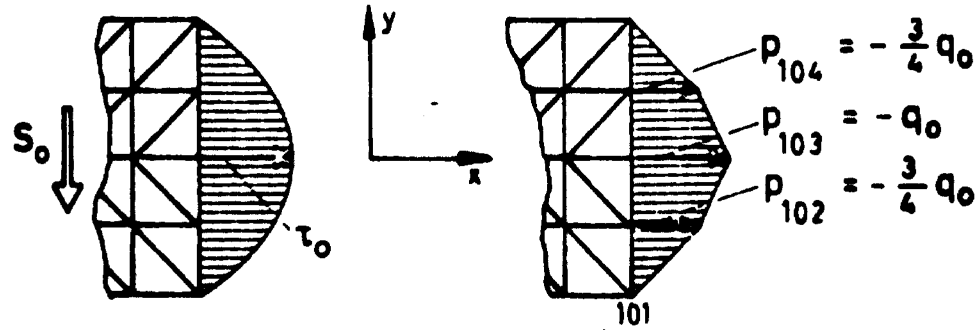
The values calculated for the various models are listed in table 6.1. A few comments thereto:
In load case 1 all elements apply equally well, a thing which was clear from the start. The load produces a constant stress everywhere in the beam, which is why it is sufficient to use the simplest element TRIM 3 which enables you to calculate exactly a uniform stress field. Please note that the exist-

ing loading pattern - i.e. the stress distribution expected - should be taken into account when selecting the elements.

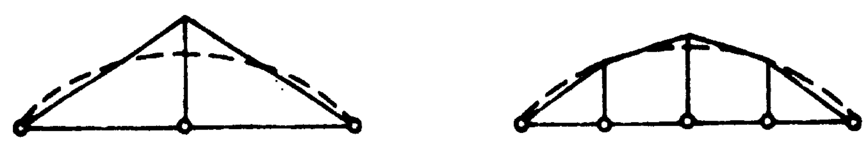
Type of element	Load case 1			Load case 2			Load case 3		
	σ_{xx}	σ_{yy}	σ_{xy}	σ_{xx}	σ_{yy}	σ_{xy}	σ_{xx} $_{y=c}$	σ_{yy}	σ_{xy}
Analytical values	12	0	0	0	0	-4.5	12	0	0
TRIM 3	12	0	0	0	0	-3.9179	5.6622	.7425	.3489
QUAM 4	12	0	0	0	0	-4.1249	8.7215	0	0
TRIM 6	12	0	0	0	0	-5.428	12	0	0
QUAM 9	12	0	0	0	0	-5.6418	12	0	0

Table 6.1 **Comparison of elements at the free end of the beam.**

In load case 2 you experience your first disappointments with FEM. This is not caused by the method being bad but by the way in which the existing shear load (i.e. the distributed load) has been replaced by corresponding nodal forces: When determining the nodal loads only one linear variation between the corner nodes was assumed.



$$\int_A \tau dA = \int_{-c}^{c} q dy = S_o = -3 \therefore q_o = 4,8 \cdot t$$



The figure above shows quite clearly that a QUAM-4 idealization gives a better approximation to the existing parabolic load than that of an equivalent TRIM-6 idealization. This should make you consider the significance of making correct comparisons. By the way: This particular fault only appears close to the nodal loads: the results obtained one element layer away from the free end are very satisfactory.

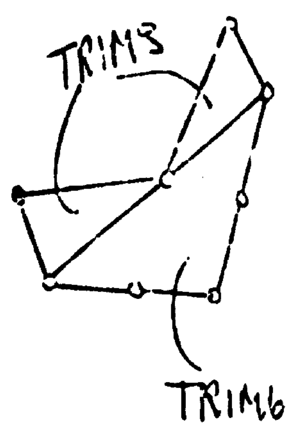
Node No.	5	15	25	35	45	55	65	75	85	95	105
Anal.	90	81	72	63	54	45	36	27	18	9	0
TRIM 6	91.87	81.51	72.78	63.76	54.76	45.76	36.76	27.76	18.76	9.74	1.55
QUAM 9	91.95	82.47	72.11	63.00	53.99	44.99	35.99	26.99	17.99	8.92	0.11

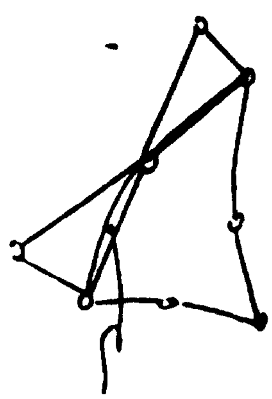
Table 6.2 Distribution of σ_{xx} in load case 2.

Fig. 6.1 shows an additional comparison, demonstrating unequivocally that merely increasing the number of unknowns in simple elements, such as TRIM 3, does not mean that the results obtained will be of the same quality as those obtained by the use of higher-order elements, such as TRIM 6. Generally speaking it is better to use one TRIM-6 element rather than the equivalent TRIM-3 model, i.e. 4 TRIM-3 elements.

Mention should also be made of the fact that similar considerations may apply to three-dimensional elements as well; cf. Fig. 6.2. Despite the fact that this example does not yield an exact value it is possible to guess at this value, basing the guess on the boundary properties of the displacement method.

Finally, permit me to warn against the combined use of different types of elements which are incompatible as far as kinematic consistency is concerned.





Incompatibility in kinematic consistency.

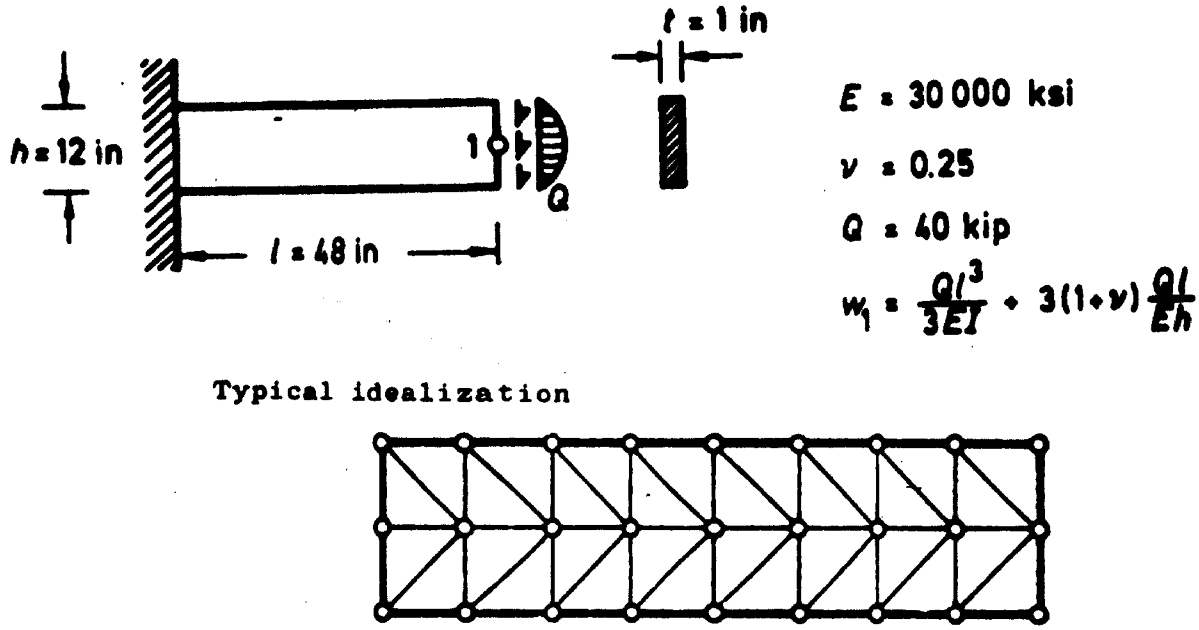


Fig. 6.1 a Cantilever beam subjected to transverse load
Data and typical idealization

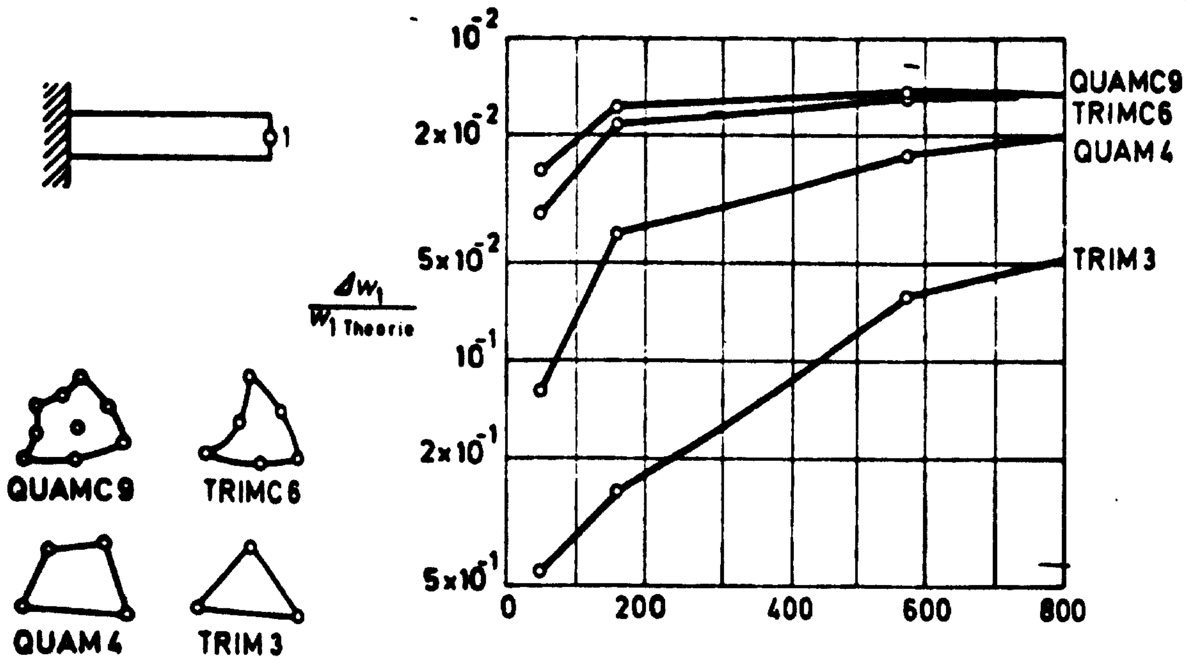


Fig. 6.1 b Cantilever beam subjected to transverse load
Accuracy of final displacement w_1 obtained by various methods of idealization

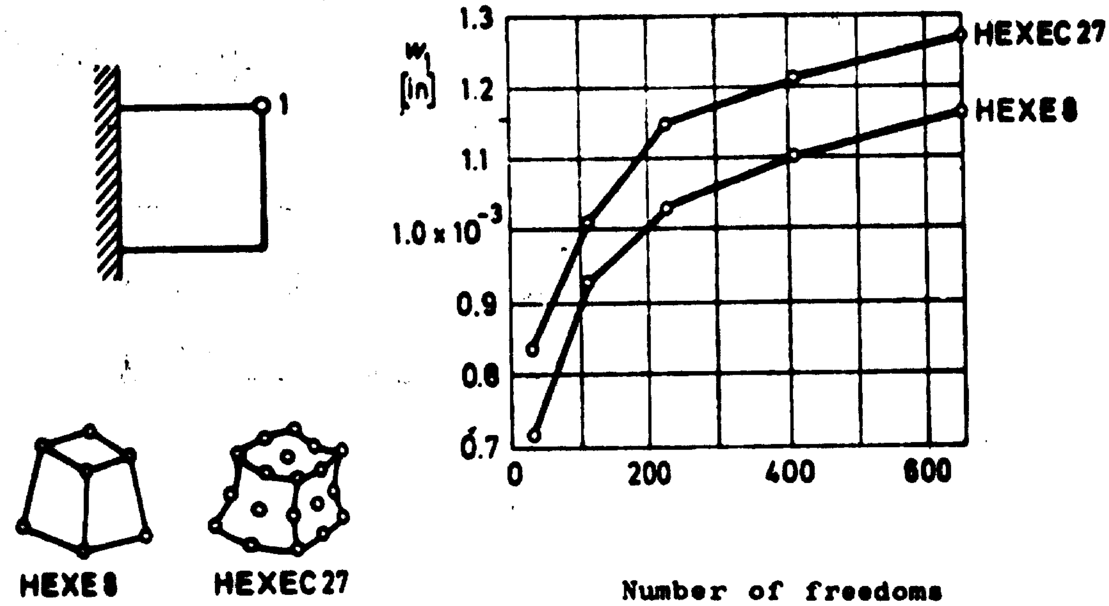


Fig. 6.2 Short cantilever support subjected to transverse load.
Final displacements w_1 obtained by various methods of idealization.

7 THE DISPLACEMENT METHOD

This chapter is intended to be a brief introduction to the displacement method. (Detailed introductions are contained in several publications other than this.)

As has been mentioned before this method implies generating an equivalent physical model of the structure by tying together various and often different elements at their nodal points. The unknowns of this model are the displacements defined by means of matrix r , incorporating all existing boundary conditions and other limitations.

Loads, such as temperature loads, manufacturing faults, and water pressure are defined at the pertinent nodes in matrix R in a kinematically consistent manner. The entries in the load matrix correspond with the selected directions of the unknowns r , so that work may be expressed as follows:

$$\text{Work} = R^t r = r^t R$$

We have seen before, in equation (4.1), how to write the linear system of equations constituting the link between R and r :

$$Kr = R \tag{7.1}$$

Let us first of all, however, consider an element whose displacement processes have been defined by its deformation mode. Kinematic nodal point displacements of this element are designated by vector ρ , and the nodal point loads corresponding to ρ are designated as P (the Greek letter, capital 'Rho'). It is obvious, then, that in each element

$$P_e = k_e \rho_e \tag{7.2}$$

Note the similarity of this equation to (7.1).

(7.2) can be expanded to take in all n-elements simultaneously:

$$\boxed{P = k\rho} \tag{7.3}$$

Obviously, in this case both $\mathbf{P}$ and $\boldsymbol{\rho}$ are hypermatrices (hyper column vectors, properly speaking),

$$\mathbf{P} = \{P_1 \; P_2 \cdots P_e \cdots P_n\}$$

$$\boldsymbol{\rho} = \{\rho_1 \; \rho_2 \cdots \rho_e \cdots \rho_n\}$$

the element stiffnesses being contained in a hyperdiagonal matrix

$$\mathbf{k} = \lceil k_1 \; k_2 \cdots k_e \cdots k_n \rfloor$$

The displacement mode allows a clear definition of the displacements occurring within the element and at its boundaries.

$$u = \varphi \rho \qquad (7.4)$$

The strains caused by this deformation can be expressed as follows (cf. also equation (2.6)):

$$\varepsilon = Du = D\varphi\rho \qquad (7.5)$$

The stresses σ are (cf. also equation (2.9)):

$$\sigma = E(\varepsilon - \varepsilon_I) + \sigma_0 \qquad (7.6)$$

with σ_0 designating any possible initial stress.

In chapter 2, mention was made of the volume forces w (cf. equations (2.1) and (2.3)), and in chapter 3.1 we designated surface forces by ϕ .

Using the principle of virtual work (cf. 3.2) we may now set up the following equations, assuming a virtual displacement $\tilde{\boldsymbol{\varrho}}$:

$$\tilde{u} = \varphi \tilde{\varrho} \qquad (7.7)$$

Equation (3.3) shows that in the case of virtual displacement the equilibrium of all external and internal work (= $\mathbf{F}$) is safeguarded; cf. also equation (3.1):

$$\tilde{\varrho}^t \mathbf{F} + \int_V \tilde{\mathbf{u}}^t \omega \, dV + \int_{F_p} \tilde{\mathbf{u}}_p^t \Phi_p \, dF - \int_V \tilde{\varepsilon}^t \sigma \, dV = 0 \qquad (7.8)$$

In this equation, index 'p' designates the part of surface F subjected to Φ . Inserting equations (7.5), (7.6) and (7.7) into equation (7.8), we have

$$\tilde{\varrho}^t \mathbf{F} = \tilde{\varrho}^t \int_V \varphi^t \mathbf{D}^t [\mathbf{E}(\mathbf{D}\varphi\varrho - \varepsilon_I) + \sigma_0] dV$$

$$- \tilde{\varrho}^t \int_V \varphi^t \omega \, dV - \tilde{\varrho}^t \int_{F_p} \varphi_p^t \Phi_p \, dF$$

However, the equations given above have to be satisfied concerning any random virtual displacement:

$$\mathbf{F} = k\varrho - S_{\varepsilon_I} - S_{\sigma_0} - S_V - S_P \qquad (7.9)$$

where

$$\boxed{k = \int_V \varphi^t \mathbf{D}^t \mathbf{E} \, \mathbf{D} \varphi \, dV} \qquad (7.10)$$

represents the element stiffness matrix.

Moreover,

$$S_{\varepsilon_I} = \int_V \varphi^t \mathfrak{D} E a_I \, dV$$

$$S_{\sigma_0} = -\int_V \varphi^t D^t \sigma_0 \, dV$$

$$S_V = \int_V \varphi^t \omega \, dV \qquad (7.11)$$

$$S_P = \int_{F_P} \varphi_P^t \bar{\Phi}_P \, dF$$

These equations (7.11) represent the procedure required to compute kinematically consistent nodal loads out of distributed loads. Should the procedure not be kinematically consistent the results obtained are sure to be faulty.

For simplicity's sake, let us define

$$S = S_{\varepsilon_I} + S_{\sigma_0} + S_V + S_P \qquad (7.12)$$

If we now compare (7.3) to (7.9) we find immediately that

$$F = P - S \qquad (7.13)$$

This should suffice to explain equation (7.3). Possibly we should stress again the fact that (7.3) merely satisfies the condition of statically consistent equilibrium at the nodal points only: it is, therefore, quite possible that we may find local discrepancies. Moreover, all matrices K_e necessarily singular, containing as they do some rigid body displacements.

The next step is to expand our element-level considerations of above to the global level. To do so, we first of all— have to establish a connection between ρ element displacements and r global displacements. For this purpose we use a simple transformation or connection matrix which, referring to an element 'e', reads

$$\rho_e = a_e r \qquad (7.14)$$

This equation can be set up for all elements in a manner similar to that of (7.2) and (7.3) above:

$$\boxed{\rho = a r} \tag{7.15}$$

Provided that we have defined the same direction of the degrees of freedom for all ρ and r , a merely contains entries reading 1 and is therefore corresponding to a Bool's Matrix.

If we now apply the principle of virtual work, designating by Q the external global nodal loads, we can set up the following equation (cf. also equation (7.13)):

$$\rho^t F = r^t Q \tag{7.16}$$

Using equations (7.13) and (7.15), we have

$$r^t a^t (P - S) = r^t Q \tag{7.17}$$

Since, however, equation (7.17) must apply to any random virtual displacement we now have

$$a^t (P - S) = Q \tag{7.18}$$

We now take into account equation (7.3)

$$a^t k a r = Q + a^t S \tag{7.19}$$

and finally, we come back to (cf. (4.1) or (7.1)

$$K r = R$$

in which case

$$\boxed{K = a^t k a} \tag{7.20}$$

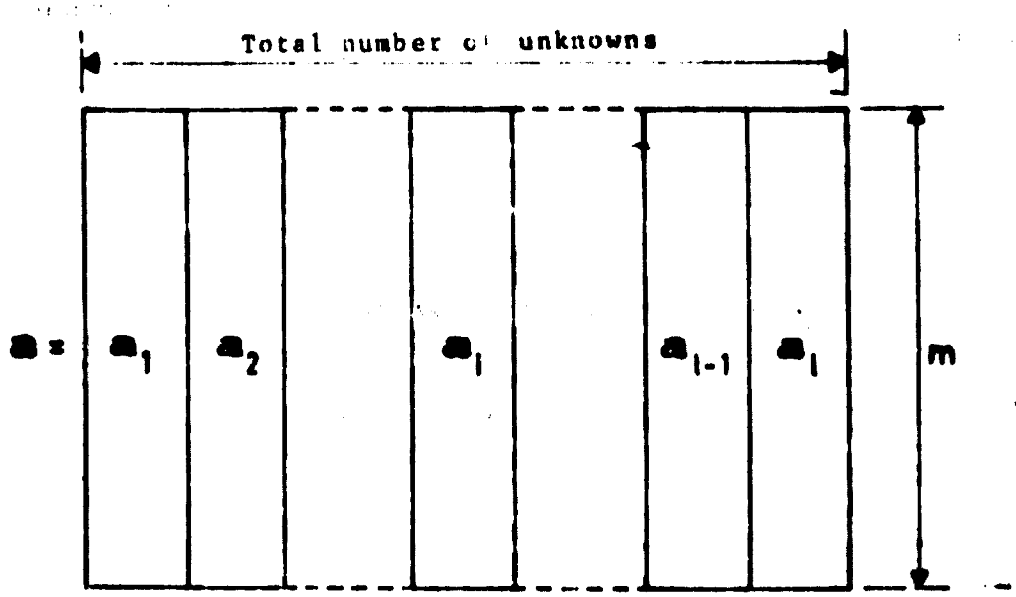
and

$$\boxed{R = Q + a^t S} \qquad (7.21)$$

The congruent transformation demonstrated in equation (7.20) is, of course, never performed as a pure multiplication of matrices in the case of large-scale problems. In such cases, algorithms (direct stiffness method) are used which permit adding the individual element entries directly into the global stiffness matrix.

$$K = \sum_{e=1}^{n} a_e^t k_e a_e \qquad (7.22)$$

In most large-scale problems, matrix $\mathbf{a}$ will be subdivided into hyper-column vectors as well.



m = sum of the global degrees of freedom of all elements.

l - hyper-column vectors of hypermatrix $\mathbf{a}$

This means that the sum of the global stiffness matrix now reads

$$K = \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \sum_{e=1}^{n} a_{ei}^{t} k_e a_{ej} \qquad (7.23)$$

Equation (7.22) applies automatically to all smaller-scale problems in which $\ell = 1$. Figures 7.1 and 7.2 indicate this procedure.

Having set up tho system of equations (7.1) we can now proceed to apply Cholesky's method to solve the simultaneous equations. (Cf. Fig. 1.4) We have to see to it, however, that matrix K is not singular: this we avoid by establishing proper boundary conditions, including suppression of rigid-body movements, which guarantees that matrix K will remain non-singular from the very start.

Having successfully solved the equation system we now know the displacements r , which enables us to determine the stresses σ by means of equation (7.6).

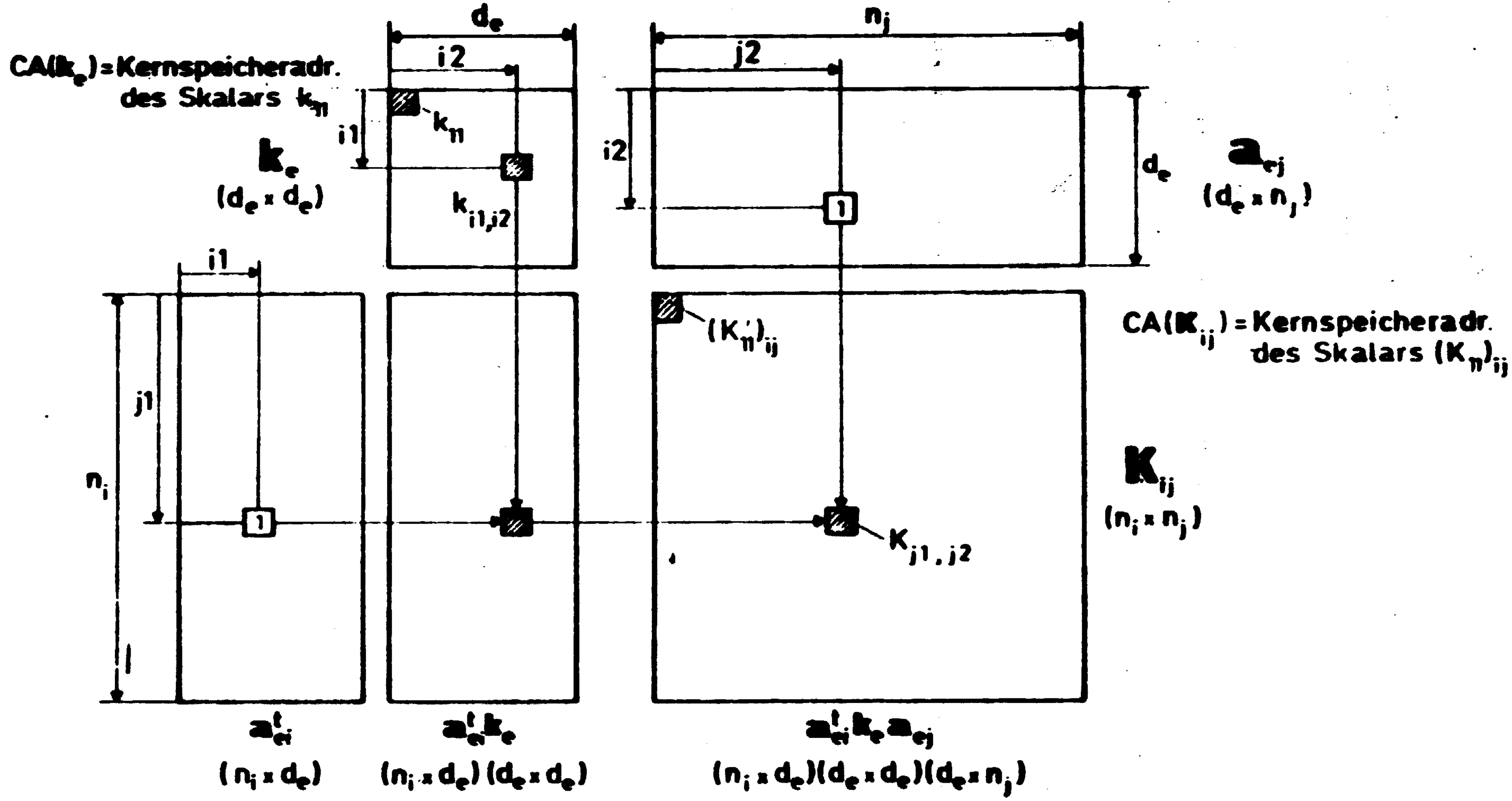
In view of the large scale of the problems which are often to be calculated these days the engineer in charge must have available an automatic substructuring-technique. It is for this reason that we shall briefly go over this procedure now.

To begin with, the displacement matrix is subdivided into two submatrices:

$$r = \{r_L \quad r_E\} \qquad (7.24)$$

with r_L containing the local freedoms, i.e. all degrees of freedom located within a substructure. r_E designates the external freedoms representing the connecting freedoms of the existing substructures. Now, the global matrix K is subdivided according to equation (7.24):

$$\begin{bmatrix} K_{LL} & K_{LE} \\ K_{EL} & K_{EE} \end{bmatrix} \begin{bmatrix} r_L \\ r_E \end{bmatrix} = \begin{bmatrix} R_L \\ R_E \end{bmatrix} \qquad (7.25)$$



$$CA(k_{i1,i2}) = CA(\mathbf{k}_e) + (i1-1) \times d_e + i2 - 1$$

$$CA(K_{j1,j2}) = CA(\mathbf{K}_{ij}) + (j1-1) \times n_j + j2 - 1$$

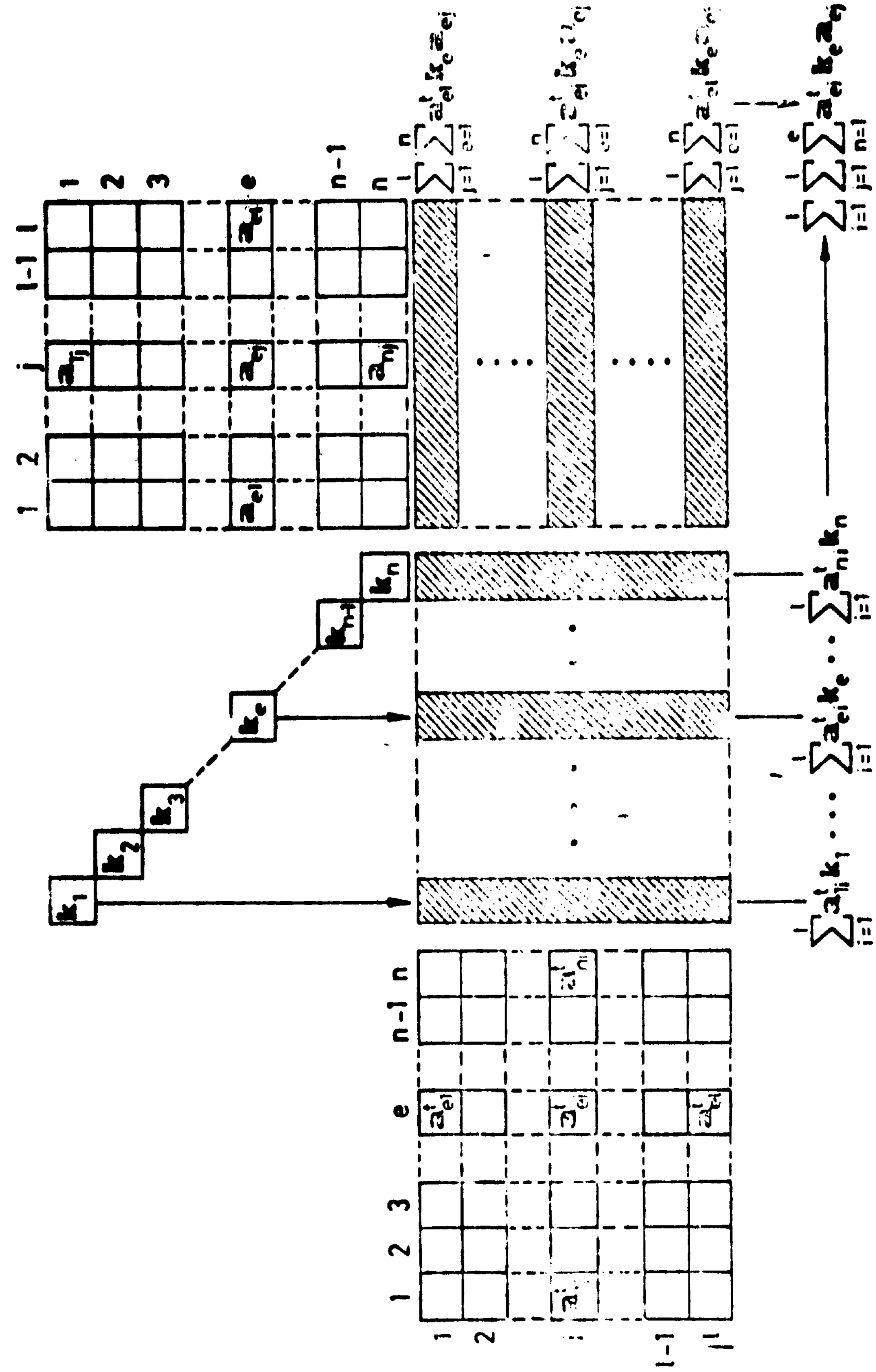


Fig. 7.2

From there, we go on to set up the following individual equations

$$r_L = K_{LL}^{-1} R_L - K_{LL}^{-1} K_{LE} r_E \tag{7.26}$$

$$(K_{EE} - K_{LE}^{t} K_{LL}^{-1} K_{LE}) r_E = R_E - K_{LE}^{t} K_{LL}^{-1} R_L \tag{7.27}$$

An alternative notation of the latter equation would be

$$\tilde{K}_E r_E = \tilde{R}_E \tag{7.28}$$

Please note the similarity between this equation and (7.1).

(7.28) shows that a substructure may easily be regarded as a super-element. Therefore it is possible to calculate the problem in several stages (recursive substructure technique).

Finally, I should like to mention the fact that prescribed displacements r_P and suppressed displacements r_S may be r_S handled in a manner similar to that shown above. The manner of subdivision used in the ASKA system is shown below:

$$Kr = R$$

$$\begin{array}{cccc|c|c}
 & & & L & r_L & \\
 & & & E & r_E & \\
 & & & P & r_P & \\
L & E & P & S & r_S = 0 & \\
\hline
K_{LL} & K_{LE} & K_{LP} & K_{LS} & R_L & L \\
K_{LE}^{t} & K_{EE} & K_{EP} & K_{ES} & R_E & E \\
K_{LP}^{t} & K_{EP}^{t} & K_{PP} & K_{PS} & R_P & P \\
K_{LS}^{t} & K_{ES}^{t} & K_{PS}^{t} & K_{SS} & R_S & S
\end{array}$$

L = Local degrees of freedom (slave unknowns)
E = External degrees of freedom (master unknowns)
P = Prescribed degrees of freedom (boundary conditions)
S = Suppressed degrees of freedom (boundary conditions)

There are many advantages to the substructuring technique. Firstly, it permits subdividing a complicated structure into easily handled components. Secondly, it allows for generating geometrically similar substructures which do not necessitate calculating the whole problem over again from the start. Thirdly, it affords the expedient, should modifications become desirable, of defining as substructures minor areas in which alterations in the complicated structure are expected to occur. In this case, only these modified structures have to be re-calculated (i.e. K_{ff}^{-1}), with all unmodified substructures already solved. In a number of cases this procedure will help to save a lot of machine time. Fourthly, it increases materially the system's general flexibility of application, especially if unusual boundary conditions, such as sliding effects within a structure, should become desirable.

8 ELASTICITY EQUATIONS IN DYNAMICS

The finite-element method is excellently applicable to dynamic problems as well. To do so, it is merely necessary to expand the linear-static equation

$$K r = R$$

correspondingly. Here, we can make profitable use of the term 'kinematically equivalent load' established in chapter 7. In accordance with d'Alembort's principle it is possible to reduce a dynamic problem to a static problem by introducing negative mass accelerations as fictitious forces. Thus, in the place of the distributed loads per unit volume we use d'Alembert's forces (cf. (3.1)).

$$w_I = -\mu \ddot{u} \tag{8.1}$$

In this case, μ represents mass density and

$$\ddot{u} = \frac{\partial^2 u}{\partial t^2} \tag{8.2}$$

represents local acceleration Applying equation (7.4), we have

$$w_I = -\mu \varphi \ddot{\rho} \tag{8.3}$$

If we now introduce equation (8.3) into equation (7.11), using S_v , we arrive at the quasi-static nodal forces to be determined

$$P_I = -\int_v \mu \varphi^t \varphi \, dv \, \ddot{\rho}$$

or

$$\boxed{P_I = -m \ddot{\rho}} \tag{8.4}$$

in which case

$$\boxed{m = \int_V \mu \varphi^t \varphi \, dV} \qquad (8.5)$$

represents the kinematic-consistent mass matrix of a sub-structure.

The dynamic equilibrium of a discretized structure can be expressed as follows:

$$R_I + R_D + R_S = R(t) \qquad (8.6)$$

In this case,

R_I = mass forces;
R_D = damping forces;
R_S = elastic forces.

Elastic forces have already been defined in equation (7.1) as

$$R_S = Kr \qquad (8.7)$$

Based on d'Alembert's Principle, mass forces may be expressed similarly to individual elements (cf. equation (8.4)),

$$R_I = M\ddot{r} \qquad (8.8)$$

with M representing the structure's global mass matrix.

Finally, in case of viscous damping the damping forces may be expressed as follows,

$$R_D = C\dot{r} \tag{8.9}$$

with C representing the global damping matrix.

With the aid of equations (8.4), (8.5), (8.6), (8.7) and (8.9), we can now express the total displacement equation applying to the entire structure to read,

$$\boxed{M\ddot{r} + C\dot{r} + Kr = R(t)} \tag{8.10}$$

It now becomes necessary to define the two matrices M and C, using the principle of virtual work.

9 THE PRINCIPLE OF VIRTUAL WORK APPLIED TO DYNAMIC PROBLEMS

To begin with, let us consider an elastic body deformed by dynamic forces. We may assume that within a specific time interval the displacement vector $\mathbf{u}_0$ will be subjected to a certain amount of virtual alteration

$$\mathbf{u}_1 = \mathbf{u}_0 + \delta\mathbf{u} \tag{9.1}$$

The virtual displacement factor $\delta\mathbf{u}$ is infinitesimal as well as consistent with the given boundary conditions of the total structure. The virtual displacement mentioned above causes consistent virtual strains, $\delta\boldsymbol{\varepsilon}$, which can be used to calculate the momentary alterations of strain energy δU_i . In a dynamic process, the external virtual work, therefore, consists of the work of the volume forces, ($\mathbf{w}$), the surface forces ($\boldsymbol{\phi}$), the singular forces ($\mathbf{P}$), and of inertia, disregarding damping as a matter of expediency. Applying the principle of virtual work, we have (cf. equation (7.8)

$$\delta U_i = \delta W_i - \int_V \mu\, \delta\mathbf{u}^t \ddot{\mathbf{u}}\, dV \tag{9.2}$$

in which case the strain energy is

$$\delta U_i = \int_V \delta\boldsymbol{\varepsilon}^t \boldsymbol{\sigma}\, dV \tag{9.3}$$

and

$$\delta W_i = \int_V \delta\mathbf{u}^t \mathbf{w}\, dV + \int_F \delta\mathbf{u}^t \boldsymbol{\phi}\, dF + \delta\boldsymbol{\varrho}^t \mathbf{P} \tag{9.4}$$

with $\delta\boldsymbol{\varrho}$ containing the virtual nodal displacements pertaining to $\mathbf{P}$ (cf. equation (7.4)).

Work must be the same, both on the element level ($\boldsymbol{\varrho}, \mathbf{P}$) and on the global level ($\mathbf{r}, \mathbf{Q}$) (cf. equation (7.16)).

$$\delta \boldsymbol{\varrho}^t \mathbf{P}(t) = \delta \mathbf{r}^t \mathbf{Q}(t) \tag{9.5}$$

We can extract $\delta \boldsymbol{\varrho}^t \mathbf{P}$ from equation (9.4),

$$\delta \mathbf{r}^t \mathbf{Q}(t) = \delta W_i - \int_V \delta \mathbf{u}^t \boldsymbol{\omega}\, dV - \int_F \delta \mathbf{u}^t \boldsymbol{\Phi}\, dF$$

extract δW_i from equation (9.2)

$$\delta \mathbf{r}^t \mathbf{Q}(t) = \delta U_i + \int_V \mu\, \delta \mathbf{u}^t \ddot{\mathbf{u}}\, dV - \int_V \delta \mathbf{u}^t \boldsymbol{\omega}\, dV - \int_F \delta \mathbf{u}^t \boldsymbol{\Phi}\, dF$$

and, finally, δU_i from equation (9.3)

$$\delta U_i = \int_V \delta \boldsymbol{\varepsilon}^t \boldsymbol{\sigma}\, dV$$

Now, it follows from equations (7.4) and (7.15) that

$$\mathbf{u} = \boldsymbol{\varphi} \boldsymbol{\varrho} = \boldsymbol{\varphi} \mathbf{a} \mathbf{r}$$

and that, if $\boldsymbol{\varphi}$ is non-variable over time,

$$\ddot{\mathbf{u}} = \boldsymbol{\varphi} \ddot{\boldsymbol{\varrho}} = \boldsymbol{\varphi} \mathbf{a} \ddot{\mathbf{r}} \tag{9.6}$$

Moreover, equation (7.5) shows that

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} = \mathbf{D}\boldsymbol{\varphi}\boldsymbol{\varrho} = \mathbf{D}\boldsymbol{\varphi}\mathbf{a}\mathbf{r} \tag{9.7}$$

Applying equations (9.6) and (9.7), we have

$$\delta r^t Q(t) = \delta r^t \int_V a^t \varphi^t D^t \sigma \, dV + \delta r^t \int_V \mu a^t \varphi^t \varphi a \, dV \ddot{r}$$

$$- \delta r^t \int_V a^t \varphi^t w \, dV - \delta r^t \int_F a^t \varphi^t \varphi \, dF$$

With δr representing a virtual displacement, we apply equation (7.6) to arrive at

$$Q(t) = a^t \int_V \mu \varphi^t \varphi \, dV a \ddot{r} + a^t \int_V \varphi^t D^t E D \varphi \, dV a r - a^t S(t) \qquad (9.8)$$

the definition of $S(t)$ being derived from equation (7.12).

If we now insert into this equation

$$\boxed{R(t) = Q(t) + a^t S(t)} \qquad (9.9)$$

in a manner similar to that employed in equation (7.21) and if, after that, we use equations (7.10) and (8.5), equation (9.8) now reads

$$a^t m a \ddot{r} + a^t k a r = R(t) \qquad (9.10)$$

In comparison to (8.7) and (8.8), an alternative notation would be

$$M \ddot{r} + K r = R(t) \qquad (9.11)$$

in which case,

$$\boxed{M = a^t m a} \qquad (9.12)$$

and, K being defined as in equation (7.20),

$$K = a^t k a$$

The damping matrix can be generated in a similar manner,

$$\boxed{C = a^t c a} \tag{9.13}$$

c representing the individual viscous damping matrices of the elements. In cases of structural damping of an actual structure, it is not always easy to arrive at the properties of matrix C . In many cases proportional damping is assumed, using

$$C = \alpha K + \beta M \tag{9.14}$$

We should mention, however, that the damping characteristics of materials and structures are still obscured by many open questions which can only be answered by intensive experimenting and research.

10 MASS MATRICES

The kinematically consistent mass matrix (often called equivalent mass matrix) of an element can be derived from the following equation (cf. also equation (8.5)),

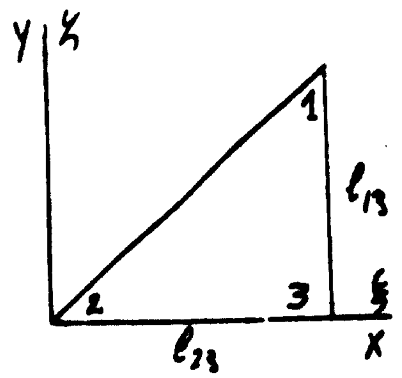
$$m = \int_V \mu \, \varphi^t \varphi \, dV \tag{10.1}$$

the relation between the displacements u within and at the boundaries of an element on the one hand and the nodal displacements ρ of the element on the other being expressed as follows (cf. equation (7.4)) .

$$u = \varphi \rho \tag{10.2}$$

Generally, dynamic processes do not have a clear matrix applicable to the entire structure. However, discretizing the structure into individual elements yields an approximation to the actual dynamic processes which may be deemed satisfactory in most practical cases.

In this chapter, we shall use one type of element - TRIM 3 - to follow the simple procedure of compiling a corresponding mass matrix. First of all, we simply assume that the element is aligned in the following fashion relative to the global axes x and y.



Using non-dimensional coordinates (ξ, η) we arrive at

$$\begin{array}{cccc} & 1 & 2 & 3 \\ \varphi_n = & [\,\eta & (1-\xi) & \xi\,] \end{array} .$$

The displacements ocurring within the element towards the direction of x, designated as u_{xn}, are

$$u_{xn} = \varphi_n g_x l_{23} l_{13}$$

with

$$g_x = \{ g_{x1} \quad g_{x2} \quad g_{x3} \}$$

Given a constant thickness t, it is now simple to calculate m

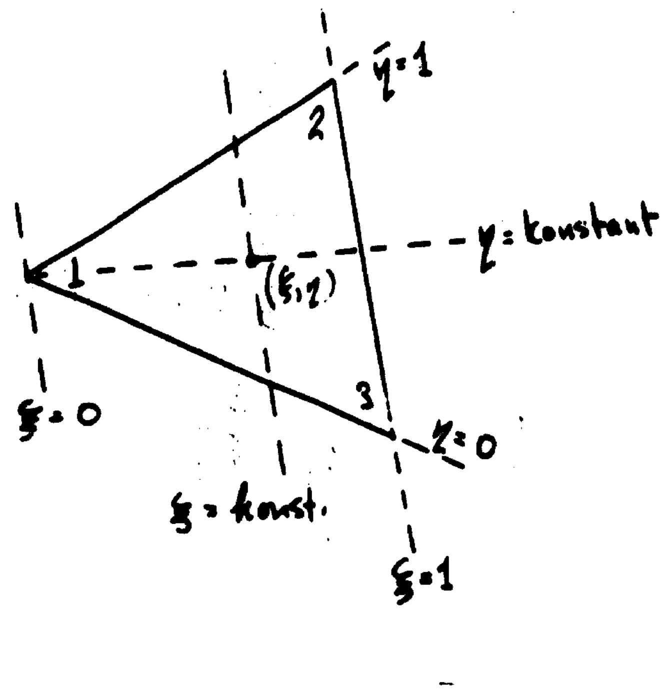
$$m_x = \mu t \int \varphi^t \varphi \, dx\, dy = \mu t \, l_{23} l_{13} \int^{1} \int^{1-\xi} \varphi_n^t \varphi_n \, d\eta \, d\xi$$

Resolving the integral yields

$$m_x = \mu t \frac{F}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

with F representing the area of the element.

Generally, however, an element's orientation relative to the global axes will not be as we have assumed above. To compensate for this, the method of polar coordinates is used in practice. This system of coordinates ξ, η is defined in the following figure.



The figure shows that, using polar coordinates, we have

$$\varphi_n = \left[(1-\xi) \quad \xi\eta \quad \xi(1-\eta) \right] \tag{10.3}$$

Furthermore,

$$\begin{aligned} x &= x_1 + \xi(x_{31} - \eta x_{32}) \\ y &= y_1 + \xi(y_{31} - \eta y_{32}) \end{aligned} \tag{10.4}$$

so that Jacobi's Transformation between the polar and the cartesian coordinates now reads as follows:

$$J(x,y) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = 2F\xi \tag{10.5}$$

Consequently, we can simply use equation (10.1), given a constant thickness t:

$$m = \mu t \int_0^1 \int_0^1 \varphi_n^t \varphi_n \left| J(x,y) \right| d\xi d\eta \tag{10.6}$$

Obviously, matrix (10.6) given above applies both to the x and y direction.

$$m = \frac{1}{12} \mu t F \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

11 VIBRATIONS AND DYNAMIC RESPONSE

In this chapter we shall deal with small-scale harmonic vibrations having a finite number of unknowns. Without damping, i.e. with $\mathbf{C} = 0$, oscillatory vibrations are bound to continue forever - an impossible thing to happen in practice, of course. These oscillatory movements occur only in certain specific frequencies, reflecting certain specific states of deformation, which is why they are often called 'characteristic modes'.

Basically, two different types of oscillation are possible:

- Free oscillations; and
- Forced oscillations.

For completeness' sake, let us repeat the total displacement equation of a global structure (8.10):

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R}(t) \qquad (11.1)$$

Generally speaking, there are two possible ways to solve this system of equations:

- Modal superposition theorem; and
- Direct integration.

The former method is often used for problems expected to involve minor amplitudes only. Thus, the displacement vector is expressed as a linear function of the characteristic mode by means of modal amplitudes. This process yields a simple uncoupled equation for each mode of the structure. After solving each individual equation the final result is arrived at by superposition. To do so, we first have to find the natural frequencies and their natural modes. The equation we need in this case is (free oscillation, undamped)

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{0} \qquad (11.2)$$

This equation expresses a simple harmonic oscillation. The displacement vector can be simply expressed as follows:

$$\mathbf{r} = \mathbf{q}e^{i\omega t} \qquad (11.3)$$

Inserting equation (11.3) into (11.2), we have

$$(K - \omega^2 M)\,q = 0 \tag{11.4}$$

thus expressing what is generally termed the 'general linear eigenvalue problem', scalar ω_i being the eigenvalue and q_i being the matching eigenvector. This last equation is also often called charateristic equation.

Often it is necessary and/or desirable, for reasons of economy and numerical handiness, to reduce the number of global degrees of freedom. This is done by subdividing the displacement vector into two distinct types of freedoms,

- master degrees of freedom r_m, and
- slave degrees of freedom r_d.

$$r = [\,r_d \quad r_m\,] \tag{11.5}$$

This method is often called the 'static condensation method'.

Corresponding to the splitting of r matrices K and M are subdivided as well.

$$K = \begin{bmatrix} K_{dd} & K_{dm} \\ \text{Sym.} & K_{mm} \end{bmatrix} \tag{11.6}$$

$$M = \begin{bmatrix} M_{dd} & M_{dm} \\ \text{Sym.} & M_{mm} \end{bmatrix} \tag{11.7}$$

Now, we presuppose that slave degrees of freedom are dependent on master freedoms.

$$r = \begin{bmatrix} T \\ I \end{bmatrix} r_m \; ; \quad r_d = T\, r_m \qquad (11.8)$$

and that the frequency equation thus reduced or condensed now reads

$$(\tilde{K} - \omega^2 \tilde{M})\, r_m = 0 \qquad (11.9)$$

with $\tilde{K}$ and $\tilde{M}$ being the condensed stiffness and mass matrix respectively. These two matrices can be found by equalising kinetic energy (KE) and strain energy (SE) of the structure concerned.

$$SE = \frac{1}{2} r^t K r = \frac{1}{2} r_m^t \tilde{K} r_m \qquad (11.10)$$

$$KE = \frac{1}{2} \dot{r}^t M \dot{r} = \frac{1}{2} \dot{r}_m^t \tilde{M} \dot{r}_m \qquad (11.11)$$

Now we use equation (11.8) to determine $\tilde{K}$ and $\tilde{M}$.

$$\frac{1}{2} \dot{r}_m^t \tilde{M} \dot{r}_m = \frac{1}{2} \dot{r}_m^t [T^t \quad I]\, M \{T \quad I\} \dot{r}_m \qquad (11.12)$$

$$\frac{1}{2} r_m^t \tilde{K} r_m = \frac{1}{2} r_m^t [T^t \quad I]\, K \{T \quad I\} r_m \qquad (11.13)$$

But, as those equations apply to all r_m we must have

$$\tilde{K} = K_{mm} + T^t K^t_{md} + K_{md} T + T^t K_{dd} T \qquad (11.14)$$

$$\tilde{M} = M_{mm} + T^t M^t_{md} + M_{md} T + T^t M_{dd} T \qquad (11.15)$$

Now we have to find the transformation matrix, T . For this purpose we assume that our slave degrees of freedom r_d are equal to those freedoms which are bound to occur in a structure not subjected to any load except in correspondence with the prescribed displacements r_m .

$$\begin{bmatrix} K_{dd} & K_{dm} \\ \text{sym.} & K_{mm} \end{bmatrix} \begin{bmatrix} r_d \\ r_m \end{bmatrix} = \begin{bmatrix} 0 \\ R_m \end{bmatrix} \qquad (11.16)$$

or

$$K_{dd} r_d + K_{dm} r_m = 0 \qquad (11.17)$$

which again gives

$$r_d = - K_{dd}^{-1} K_{dm} r_m \qquad (11.18)$$

If we compare this to equation (11.8) our $\mathbf{T}$ now reads as

$$\mathbf{T} = - K_{dd}^{-1} K_{dm} \tag{11.19}$$

If we now combine equations (11.12) and (11.14) we have

$$\boxed{\tilde{K} = K_{mm} - K_{dm}^{t} K_{dd}^{-1} K_{dm}} \tag{11.20}$$

Note the resemblance to equation (7.27). The equation given above shows that vector $\mathbf{T}_d$ may be regarded as an internal degree of freedom. The condensed mass matrix was defined in (11.15). We should note, however, that although the static condensation of the global stiffness matrix $\mathbf{K}$ represents a mathematically exact procedure the corresponding process of condensing mass matrix $\mathbf{M}$ necessitates making additional assumptions concerning displacements.

When applying direct integration to solve equation (11.1) it is not necessary to solve the eigenvalue problem, a process which obviously, under certain circumstances, is apt to consume a lot of machine time. Moreover, this method is applicable to non-linear problems as well. This method implies splitting up the response process by finite differences in time. The response at the end of the first finite step is calculated on the basis of the initial situation as well as the load during the first step. The results thus obtained at the end of each step are then used as an initial basis for calculating the subsequent step. Describing the entire range of methods, each with its specific advantages and with a following of advocates, would go beyond the scope of this paper.

12 NON-LINEAR PROBLEMS

Generally speaking there are two kinds of non-linear problems,

- non-linear material behaviour (elasto-plastic phenomena), and
- non-linear geometrical phenomena.

Problems combining both kinds of phenomena still present us with well-nigh insurmountable obstacles in solving practical engineering problems.

Again, we solve these problems by splitting them up into small steps, each step presupposing a linear process; this is, in other words, an iterative approach.

Moreover, let me mention the fact that the superposition theorem is not applicable to non-linear problems, which means that if we have several load cases we have to deal separately with each individual global load case.

The scope of these brief remarks does not afford an oppotunity to deal thoroughly with these non-linear processes. I shall merely attempt to give a brief survey.

Non - Linear Material Properties

If we are dealing with non-linear material properties we merely have to modify linear equation (7.6). We repeat this equation below as a reminder.

$$\sigma = E(\varepsilon - \varepsilon_I) + \sigma_0 \qquad (12.1)$$

A general non-linear stress-strain relation can be formally expressed as follows:

$$F(\sigma, \varepsilon) = 0 \qquad (12.2)$$

As the compatibility equation (7.5) is applicable here, we havo

$$\varepsilon = Du = D\varphi\rho \qquad (12.3)$$

to take into account as well as the requisite conditions of equilibrium. Its is obvious, therefore, that we shall find the solution of the non-linear problem (12.2) provided a) that we modify one or more of the matrices E, σ_0 and ε_I of equation (12.1) and b) that we can find a solution to equation (7.1)

$$K r = R \tag{12.4}$$

in which the stresses σ and strains ε obtained will satisfy equation (12.2).

To be on the safe side it should be mentioned that there is virtually no theorem guaranteed to provide an exact or correct solution to a non-linear problem. It is therefore perfectly possible to obtain incorrect results in spite of the fact that all necessary conditions, such as equilibrium, displacement consistency, and a correct stress-strain relation were fulfilled.

To obtain a solution it is always necessary to employ an iteration method. According to whichever matrix, E, σ_0 or ε_I, is modified, the iteration process is either called

- Method of tangential stiffness (E), or
- Method of initial strain (ε_I), or
- Method of initial stress (σ_0).

The tangential stiffness method is applicable to all elasto-plastic problems. The matrix is generated in a manner similar to that described in chapter 7 in connection with the elastic stiffness matrix (equation 7.10).

$$k_T = \int_V \varphi^t D^t F D \varphi \, dV \tag{12.5}$$

In this case, F represents the pertinent elasto-plastic material properties which a user of ASKA, for example, would have to define beforehand, Then, we simply have

$$K_T = a^t k_T a \tag{12.6}$$

This method necessitates calculating the tangential stiffness matrix anew at every step and, far worse than this, it also necessitates solving another system of linear equations every time. Although application of the substructure technique, with only some parts of the structure showing plastic behaviour, renders the entire process much more economical there is general agreement that the initial load method (i.e. initial strain or initial stress method) is more favourable.

The initial load method may be expressed as follows (cf. (7.21) and (12.4)):

$$R_{\Delta} = K_E\, r_{\Delta} + a^t J_{\Delta} \qquad (12.7)$$

with Δ , of course, representing an increment of the vector. Matching initial loads J_{Δ} can then be used to simulate any modification of the elastic stiffness matrix. In dealing with elasto-plastic problems we now use equation (12.7). In this case, K_E is resolved only once. We merely calculate the extent of plastic strain at each step.

Using initial loads implies certain difficulties. The increment of each load vector J_{Δ} must be derived from the plastic strain increment which in turn is derived from R_{Δ} . On the other hand it is impossible to calculate plastic strain without knowing the stress increment. To solve this dilemma we have the two methods of initial strain and initial stress.

Initial stress has an advantage over the initial strain method in that it is applicable to calculating ideal plastic processes as well; otherwise, both methods are more or less equal. (Ideal plastic processes can also be dealt with by means of the tangential stiffness matrix method, by the way.)

N o n - L i n e a r G e o m e t r i c P r o c e s s e s

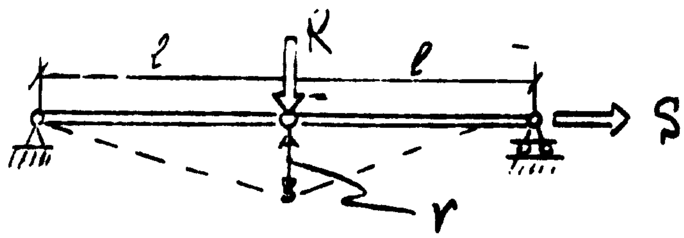
This term describes all processes in the course of which the geometry of a structure changes under load to an extent which rules out the assumption of an equilibrium existing in the deformed structure, implying that there is a non-linear relation between stresses and strains. With each iterative step it is therefore necessary to formulate anew the conditions of equilibrium; this despite the fact that the scope of strains continues small and that there is no presumption of linear material behaviour. Processes presuming small-scale strain and linear material behaviour are often called <u>'large-scale displacements'</u>.

At each iterative step of a large-scale displacement the non-linear stress-strain relation causes a change in the element matrices k . Modification of the element stiffness matrix is designated by k_G . We can, therefore, express the total matrix k as follows:

$$\boxed{k = k_E + k_G} \qquad (12.8)$$

with k_E representing the elastic portion of the matrix (cf. equation (7.10)). The matrix k_G is often called the geometric stiffness of an element. This matrix is not merely dependent on the geometry but is also a function of the stresses within the element.

A simple example:



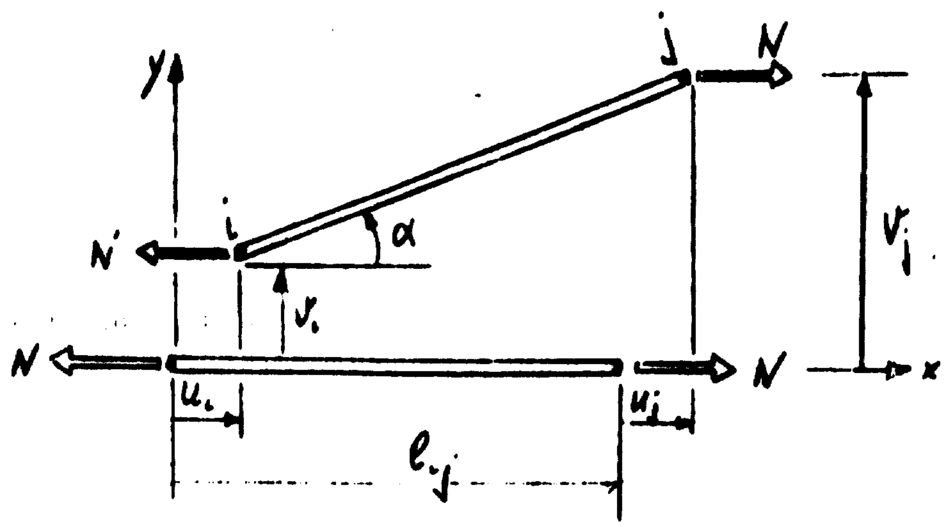
Two rods are connected by a joint. Clearly, there is no stiffness in the direction of the load R while the structure is still in its initial position, i.e. with the rods horizontal. Equilibrium will only be restored after a vertical displacement r has taken place.

$$R = \frac{2S}{l} r = K_G r$$

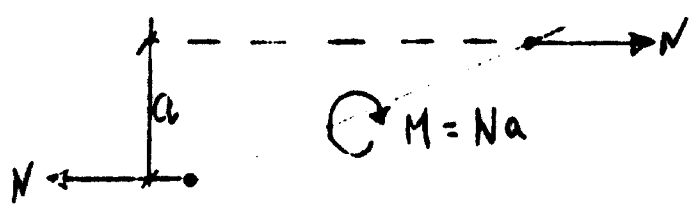
Geometric stiffness, therefore, is

$$K_G = \frac{2S}{l}$$

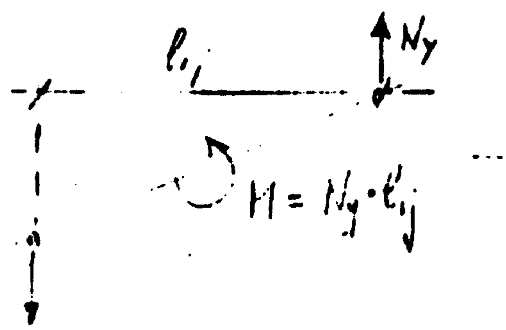
The figures shown below illustrate another example. A flange element is subjected to a perpendicular force N. Under this load, additional large-scale displacements are forced transversely to the flange (v_i, v_j).



In large-scale displacements, the force will at first continue in its original direction along the x axis. This gives rise to a moment Na.



This moment must be balanced.



Of course, both moments must be equal.

$$N a = N_y \ell_{ij}$$

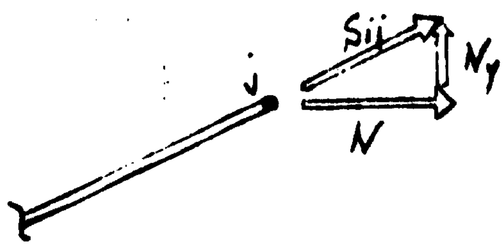
Now we have

$$a = \ell_{ij}\, tg\,\alpha = v_j - v_i$$

and, therefore,

$$N_y = \frac{1}{\ell_{ij}} (v_j - v_i)$$

If we now add the vectors of N and Ny the load is again exerted along the flange. Equilibrium is restored.



Let us now consider the linear-elastic processes, i.e. the small-scale strains. Strain along the flange is expressed as

$$\varepsilon_{ij} = \frac{\Delta \ell_{ij}}{\ell_{ij}} = \frac{(u_j - u_i)}{\ell_{ij}}$$

the corresponding force being A, with the flange area remaining constant.

$$S_{ij} = A E \varepsilon_{ij} = \frac{AE}{\ell_{ij}} (u_j - u_i)$$

Now we can establish the entire stiffness matrix.

$$k = k_E + k_G$$

$$\underbrace{\begin{bmatrix} N_{xi} \\ N_{yi} \\ N_{xj} \\ N_{yj} \end{bmatrix}}_{N} = \underbrace{\frac{AE}{l_{ij}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{k_E} \underbrace{\begin{bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{bmatrix}}_{u} + \underbrace{\frac{S}{l_{ij}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{k_G} \underbrace{\begin{bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{bmatrix}}_{u}$$

or else, expressed as matrices,

$$N = (k_E + k_G)u = ku$$

Generally speaking it is possible to compile a geometric element stiffness for each element. Knowing both k_E and k_G the element is subjected to a small finite displacement increment. Now, the load vector P_Δ can be expressed as (cf. equation (7.3)):

$$P_\Delta = k\rho_\Delta = (k_E + k_G)\rho_\Delta \qquad (12.9)$$

Moreover, virtual work can be expressed to read

$$r_\Delta^t R_\Delta = \rho_\Delta^t P_\Delta$$

According to equation (7.15),

$$\rho_\Delta^t = r_\Delta^t a^t$$

so that

$$R_\Delta = a^t P_\Delta = a^t(k_E + k_G) a r_\Delta = K r_\Delta$$

(12.10)

$$\boxed{R_\Delta = K r_\Delta}$$

This equation is a mere approximation, but it grows more accurate the smaller both R_Δ and r_Δ are. In large-scale displacement problems it is unfortunately necessary to compile anew the global stiffness matrix and to solve the linear system of equations later. It is very difficult to make any reliable statement concerning the scale of the load increments. Each problem must be considered individually to ensure that its peculiarities are taken into account, always remaining sensitive to practical considerations.

In conclusion, I should like to remark that the large-scale displacement method can also be used to solve stability problems.

C-723

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