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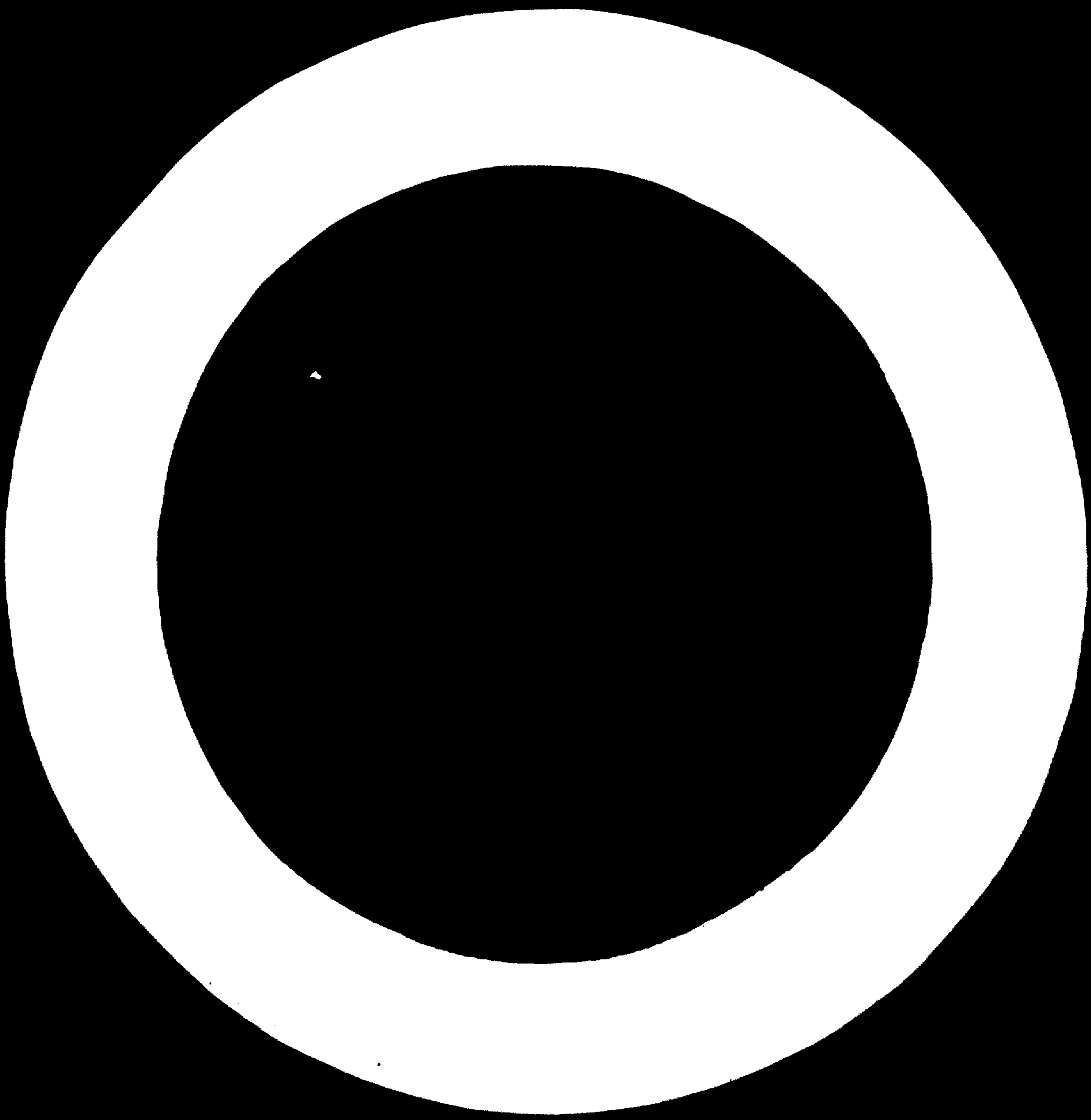
Vienna International Centre, P.O. Box 300, 1400 Vienna, Austria
Tel: (+43-1) 26026-0 · www.unido.org · unido@unido.org



Decentralization and Project Evaluation Under Economies of Scale and Indivisibilities[1]

By THOMAS VIETORISZ

Introduction

Some of the key techniques of decentralization, particularly in free-enterprise and mixed economies but increasingly also in centrally planned economies, are based either directly or indirectly on the notion of market equilibrium. The twin assumptions of *non-convexity* and the *absence of externalities* play a fundamental role in establishing the important theorems of contemporary neoclassical economics concerning the existence, stability and attainment of such equilibria, the existence of a price system and its role in decentralized adjustment, and the welfare significance of the outcome of these adjustments.[2]

The objective of this paper is to inquire into the effect that non-convexities have on the practical possibilities of decentralization by means of a price system or by other related methods. Current economic theory offers little enlightenment in this respect even though significant non-convexities are known to be present in most economic systems, particularly in the technology and organization of production, and in the field of urban land use. As most market theorems break down under such conditions it is natural to inquire not only into the existence of substitute adjustment mechanisms but also into the puzzle of how the existing decentralized pricing and market systems are capable of operating despite their admitted inefficiencies.

Economic equilibrium versus non-convexity

The essential features of economic equilibrium were put forward by the classic economists as an explanation of the

[1] The first version of this paper was presented as Discussion Paper No. CID/IPE/B.28, United Nations Inter-regional Symposium on Industrial Project Evaluation, Prague, Czechoslovakia, 11–29 October 1965, under the title "Project Evaluation in the Presence of Economies of Scale and Indivisibilities". The revised version included here was presented to the Econometric Society Meetings, New York, in December 1965.

[2] Koopmans (1957), page 25, writes (see bibliography at end of this treatise for this and other references cited herein): "The principal reason for making a convexity assumption lies not in its degree of realism but in the present state of our knowledge. When one examines the main contents of received theory of resource allocation and competitive markets it is found that its propositions depend essentially on convexity assumptions with regard to both production possibilities and preference structures."

Convexity assumptions also play a crucial role in the convergence of the characteristics of the gradient method, one of the algorithms available for solving resource allocation problems formulated in terms of mathematical programming models, and commonly held to reflect the essential features of the adjustment mechanism of the market. Arrow and Hurwicz (1960), page 87, give the following summary of these problems (note: concave functions yield convex point sets): ". . . the absence of concavity conditions on the functions involved has two consequences for the characterization of maxima (constrained or unconstrained): the first-order conditions do not completely distinguish maxima from other stationary points, and in any case do not in any way distinguish global from merely local maxima . . . no variation of the gradient method, which is based on moving uphill as measured by solely local variations, can be expected to ensure arrival at the highest of several peaks; at best, only convergence to a local maximum can be expected." The role of externalities in economic equilibria has recently been investigated in depth by Whinston (1962). On centralization versus decentralization, the role of the price system, and welfare implications, see also Arrow (1951), Tinbergen (1954), Arrow, Hurwicz and Uzawa (1958), Marschak (1959), Hurwicz (1960 a, b), Tinbergen (1962).

DR. THOMAS VIETORISZ *is Professor of Economics and Director of the Center for Economic Planning of The New School for Social Research, New York, United States, and has been associated with the United Nations as a regional expert in industrial development programming with the Economic Commission for Latin America (ECLA). His contributions to earlier* Bulletin *issues include a "Preliminary Bibliography for Industrial Development Programming", published in Nos. 5 and 6 of the* Bulletin, *and "Programming Data Summary for the Chemical Industry", published in* Bulletin *No. 10. Professor Vietorisz has also prepared studies for several United Nations meetings, including the Inter-regional Conference on the Development of Petrochemical Industries in Developing Countries held in Tehran, Iran, in November 1964, and the Inter-regional Symposium on Industrial Project Evaluation held in Prague, Czechoslovakia, in October 1965.*

behaviour of actual markets under free enterprise. Later, as the shortcomings of the market mechanism—monopoly elements, limited effective demand, unsatisfactory distribution of income and wealth, frustrated growth—became historically more important and theoretically more widely recognized, market equilibrium was still held up as an idea that could be approximated in practice to a "workable" extent. Lately, with the advent of mathematical programming techniques, it has become possible to isolate economic equilibrium from the behaviour of actual markets, and either to replace actual market behaviour or to simulate it by electronic computer solutions to planning models with varying degrees of centralization or decentralization. In fact, economic equilibrium can be adapted by means of computer solutions to models representing economic situations that even ideally competitive markets would be unable to realize; for example, multi-period resource allocation models with imposed terminal conditions[3] or with institutional limits set on the variation of prices, on resource utilization, or on activity scales.

The existence of such models and the possibility of solving them numerically do not imply that the entire economic process can be or soon will be replaced by a single large centrally solved planning model. Decentralized decision-making and, in the realm of management and planning, the multi-level organization of decision systems are essential for reasons which include the following purely economic considerations:

(1) Technical alternatives are difficult to formulate over a sufficiently wide range of factor prices for a model.

(2) It is inefficient to formulate in detail alternatives that will not be used; for this reason it is desirable that the compilation of information should alternate with analysis stage by stage. This process can be carried out most effectively near the sources of technical information in individual firms or individual sectors of the economy.

(3) The structure of a large model cannot be intuitively grasped, and therefore its blind application is hazardous; this difficulty can be overcome by co-ordinating a number of smaller models.

(4) Plan formulation must take into account the modes of execution: this requires familiarity with technical detail available only near the operating levels.

(5) Plans have to be readjusted to changing circumstances in the course of execution. Many of these changes show up at or near the operating level; thus planning capability at lower levels facilitates efficient adjustment to such changes.[4]

It has long been known that economic equilibrium, whether embodied in the postulated operation of actual markets or in the adjustments of a mathematical programming model, has inherent limitations that cannot be overcome by minor modifications of the principles upon which a market equilibrium operates. One example is the problem of fixed costs which lead to diminishing average costs as the scale of production increases. It is impossible to reconcile the requirements of (*a*) efficient resource allocation as embodied in marginal-cost pricing rules with (*b*) the need for covering the fixed costs incurred by the firm out of revenues obtained from product sales. The *exact* reconciliation of these conflicting requirements is possible only when the average cost curves of individual firms are U-shaped, and then only at selected lattice points along the quantity axis;[5] at in-between quantities either requirement (*a*) or (*b*) must be violated. Industry supply may, however, be satisfactorily *approximated* by a continuous function if the separation between lattice points is small in relation to total industry production, i.e., when there are a large number of small firms. This is the assumption of the received theory of competitive supply.

The presence of fixed costs is a case of mathematical non-convexity leading to economies of scale.[6] Such economies of scale can also occur in the absence of actual fixed costs, depending on the shape of the production function.[7] Other significant cases of non-convexity are:[8]

Indivisibilities: the necessity of planning in multiples of standardized production units: zero-one decisions on transport investments, hydroelectric projects, etc.;

Pre-emption of land area: the fact that a given plot of ground (e.g., in a densely occupied zone) has to be assigned in a zero-one fashion to individual uses;[9]

Either/or type constraints on feasible policy alternatives, prescribed sequencing of activities, etc.[10]

A decentralized decision-making system based on *linear* decentralizing instruments (master prices, administratively

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[3] The so-called "dynamic invisible hand" theorem (Dorfman, Samuelson and Solow) (1958), page 319, that extends the principle of social efficiency of perfectly competitive markets from a static to a dynamic context guarantees only that such a system, once locked on an efficient growth path, will stay on it; it cannot direct the system toward a growth path that satisfies exogenously determined terminal social objectives.

[4] Clopper Almon (in Dantzig, 1963, pages 462–465), and Victorisz (1963) discuss some of these points in relation to multi-level management and planning organizations.

[5] When the average cost curves have horizontal minimum ranges, the lattice points broaden to equilibrium ranges of finite width.

[6] A point set S is convex if the following holds: if $X_i \epsilon S$ and $\lambda_i \geqslant 0$ and $\Sigma_i \lambda_i = 1$ then $(\Sigma_i \lambda_i X_i) \epsilon S$, where $i = 1, \ldots, n$. Applied to an available technology consisting of a collection of projects this concept of convexity means that any *weighted average* of technically feasible individual projects will also be technically feasible. Note that where economies of scale are present convexity breaks down. For example, if the actual capital input requirements of a process comprise a *fixed* input plus an input proportional to scale, then *two half-sized* projects using this process will actually use more capital than *one full size* project; i.e., the average of two half-sized projects (with equal weights) will underestimate capital requirements and will thus describe an infeasible technology.

[7] Economies of scale often are expressed by an input function of the form:

$$(y/\bar{y}) = (x/\bar{x})^f,$$

where y and $\bar{y}$ are inputs corresponding to scales x and $\bar{x}$; the barred quantities are constants; and f is a constant exponent in the range $0 \leqslant f \leqslant 1$.

[8] Victorisz (1964).

[9] Koopmans and Beckman (1957).

[10] Dantzig (1960).

determined planning prices, incentive systems with linear structure) is inherently unable to guarantee attainment of an optimal equilibrium position unless all sources of non-convexity—such as fixed costs and others—are either absent or rendered inoperative by special circumstances which occur in competitive supply. Therefore no decentralized decision criteria based on the notion of economic equilibrium and involving correspondingly a linear version of pricing or incentive systems—whether these be market prices, corrected opportunity costs, electronically computed shadow prices based on mathematical programming models, or administratively fixed prices in a planned economy[11]—can be relied upon in the presence of non-convexities. The criteria may yield acceptable results, but they also can result in gross misallocations.

Two illustrations will indicate the kinds of market outcomes that are possible when linear decentralizing instruments are used in the presence of non-convexities. Chenery (1959) constructs a detailed numerical example of steel production and iron-ore mining with strong economies of scale in a developing country. The analysis reveals that either one of these two activities is profitable only when the other is present. Thus a decentralized decision system based on profit (or social marginal product) misses an attractive *joint* investment opportunity. When neither of these activities is yet established the decentralized decision maker studying an activity in isolation will decide that it is unprofitable; thus neither of the two activities can precede the other and the profitable complex of the two activities will never be attained.[12] Koopmans and Beckman (1957) construct an example which shows non-convexities involved in the assignment of productive activities to particular locations that cannot be shared between activities. For example, in a urban area a given block or plot of land cannot be used for *both* a large shopping centre and an industrial plant. Thus the present location of an activity will affect the costs of all other activities in such a way that, with *any* locational pattern, incentives will exist for some producers to change their locations, and the possibility of a stable equilibrium price system is negated. In many locational problems no assignments are required; for example, if locations have to be chosen for industries that can be located at several regional centres at large distances from each other, the land requirements at these centres will usually be very small in comparison with the available industrial sites and thus several activities may easily be located at the same centre. This kind of locational problem is generally convex unless economies of scale occur independently in the production or transport activities. A stable price system can be utilized in the usual way for the definition of project evaluation criteria.

When significant non-convexities are known to be operating—important industrial processes whose optimal scales of operation are higher than the level of demand of a small country, important decisions concerning investments in transport arteries, etc.—the only reliable method of taking detailed decisions is a comprehensive analysis of all alternatives within the framework of a mathematical programming model in which non-convexities are explicitly accounted for.

Integer programming is the analytical tool of choice in the formulation of such models. A wide variety of all the non-convexities in the field of economic planning can be represented or approximated adequately by integer programming models.[13] In these models, some or all variables are restricted to integer values instead of being allowed to vary in a continuous fashion. Although exact solutions to such problems are often difficult to obtain (except for small problems), there are several methods which, in combination, can be employed to obtain good sub-optimal solutions as well as the upper bounds on the possibility of further improvement; thus the exact solution values can be approximated subject to a known margin of error.[14]

All that has been said earlier about the essential role of decentralization in economic decision-making is equally valid for convex and non-convex systems. Even if they become amenable to rapid and exact numerical solution, large integer programming models can never replace the entire economic process. Thus the fact that non-convexities can be adequately handled by certain mathematical models is insufficient; it is indispensable that at least an approximate inquiry be made into the possibilities of decentralization and multi-level decision-making in systems represented by such models. This problem will be solved in two stages in the present article. First, a two-level *linear* decision model will be analysed graphically; next, fixed costs will be introduced, making the model non-convex, and parallels will be drawn between decentralization possibilities in the linear and the non-convex cases. The non-convex decision model will then be used to explore the relationship between average and marginal costs and the degree of indivisibility in a system, and also to shed new light on the relationship between non-convexities and externalities.

The decomposition principle in linear systems

Some of the phenomena that occur in multi-level decision-making or planning systems can be analysed by

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[11] Kornai and Liptak (1962) discuss different kinds of profitability indexes used as decentralizing instruments in a centrally planned economy.

[12] There have been numerous qualitative discussions of the interrelations between industries in the course of economic development due to economies of scale and externalities. Economies of scale create technical interrelations such as discussed by Chenery (1959), they also lead to complementarity between industries producing consumer goods. External economies arise in education, labour training and activities aimed at securing technical progress; in social-overhead investments (transport, energy, communications); in housing and urban facilities; in government and other public services. See for example Rosenstein-Rodan (1943, 1961), Hirschman (1958).

[13] Dantzig (1960), Vietorisz (1964).

[14] Recent surveys of integer programming will be found in Balinski (1965) and Beale (1965). Gomory (1963) summarizes the relationship between large and non-convex linear programming models; Gomory (1965) gives an appraisal of *rounded* continuous solutions and a new algorithm.

means of the *decomposition principle* developed originally for the solution of structured linear programming models.[15] Figure I indicates schematically the relationship between a two-level planning organization and the structure of a corresponding decomposition model. In the model, non-zero technical coefficients appear only within the shaded blocks (figure I(*b*)); these coefficients fall into two broad groups. First, there are the coefficients of the *special resources* of each sector. The special resources of each sector can have non-zero coefficients only in the activities of their own particular sector. Second, certain resources may have non-zero coefficients in any sectoral activity; these are designated as *connecting resources*. In addition to the sectoral activities that form the columns of figure I(*b*) there is also a column designated as *exogenous* (first column). While it is assumed that the scale at which each sectoral activity can be carried out is variable, the scale of the exogenous column is fixed. This column usually contains the given total supplies and demands of each resource. The problem is to find a programme (i.e., a combination of activity scales) which is consistent with the fixed resource supplies and demands, and which is in some sense efficient. Efficiency is defined in terms of maximizing the output or minimizing the input of a chosen connecting resource.

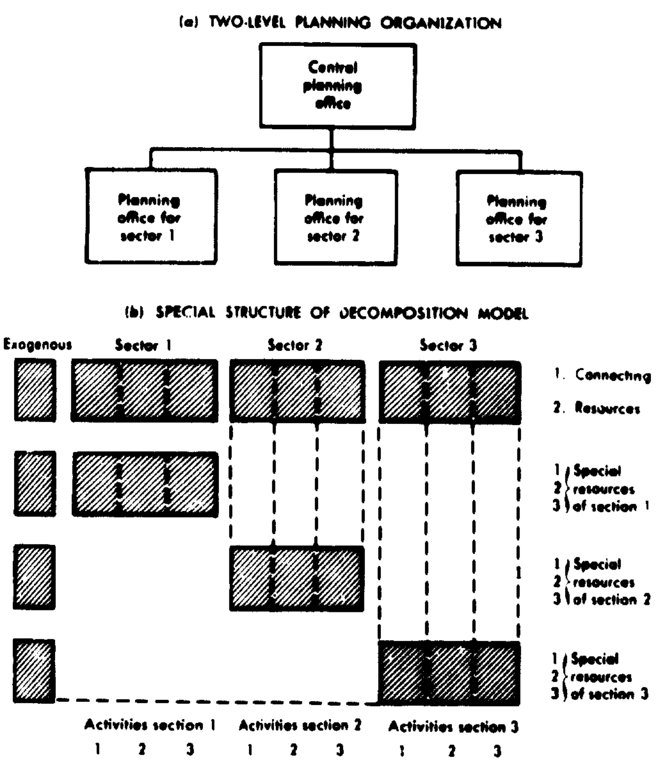

Figure I

MULTI-LEVEL PLANNING AND STRUCTURE OF DECOMPOSITION MODELS

In such a structured model the consistency and efficiency oriented decisions concerning the connecting resources correspond to the upper level of a two-level decision-making organization such as the company-wide policy committee of a multi-divisional corporation or the central economic policy body (cabinet, central planning board) of a country. The same kind of decisions concerning sectoral resources correspond to the divisional level of corporations or to the ministerial (or regional) level of entire economies. The activities of the model may represent lines of business, individual processes, or other individual technological alternatives within a division of a company or industrial branches, enterprises, or projects within a sector (or region) of an economy. A *programme* or collection of activity levels corresponds to a complete set of tentative decisions (or plans) for the entire system, subject to later confirmation and adjustment. The structure in figure I(*b*) is *angular decomposable* and represents the simplest possible relationship between the connecting and sectoral parts.[16]

Table 1 is a numerical example of a decomposition model.[17] The model has two sectors, with two special resources in each and two connecting resources, capital and labour. There are four possible activities in each sector; the scales of these activities are variable and are designated by $X_1 \ldots X_4$ for sector 1, $X_5 \ldots X_8$ for sector 2. All numerical data obey the following *sign convention:* outputs or supplies are positive, inputs or demands are negative. Thus the capital and labour coefficients of all activities are negative (inputs); there are, however, exogenous supplies of these two factors, amounting to 350 units in the case of capital, and 2,000 units in the case of labour. Once the scales of all activities are chosen in formulating a trial programme, the flows of all resources can be determined, and their balance verified. The difference between (*a*) all outputs and exogenous supplies of a resource (positive signs) and (*b*) all inputs and exogenous demands (negative signs) is defined as the *surplus* of the resource. If the surplus is zero, there is an exact balance; if positive, the resource is redundant; if negative, there is a bottleneck. In this problem, the criterion of the efficiency of a plan is economy in the use of capital; this is expressed by maximizing the surplus of capital. This formulation may be interpreted as follows: assuming that 350 units represent the limit of capital stock which can be built up by saving and foreign borrowing, the criterion of efficiency is to reduce as much as possible the need for this saving and borrowing by decreasing capital inputs. At the same time plan consistency requires that prescribed demands be met while keeping within available resource

[15] Dantzig and Wolfe (1961); also Dantzig (1963), Gomory (1963) Kornai and Liptak (1965).

[16] Entire economies described by input-output models tend to approximate the slightly more complex block-triangular structure. The mathematical properties of such systems have been analysed by Dantzig (1963). Multi-level decision or planning systems may also be described by models in which the connecting and sectoral resources do not form mutually exclusive classes but in which resources subject to upper-level decision are defined by the aggregation of sectoral resources. The logic of this kind of system has been described qualitatively (UNECAFE, 1961), but has never been subjected to exact analysis.

[17] The coefficients of this model have been based (with some necessary changes and additions) on a small illustrative model used by Chenery (1958), table 2. Fixed-cost coefficients have been added; they are not used in the linear version of the model.

supplies; these conditions can be simply expressed as the avoidance of resource bottlenecks.[18]

The model also determines the shadow prices of all resources. The price of capital is chosen as the *numeraire* resource whose price is set to unity and in terms of which other prices are expressed. The revenue (positive sign) or cost (negative sign) of a resource can be determined once the shadow prices are given and the technical coefficients of an activity are multiplied by these shadow prices. The difference between revenues and costs is the profit for any activity (variables in top margin). The dual problem consists in choosing shadow prices Y so as to minimize profits π_0 on the exogenous activity while profits on all other activities are eliminated (see annex).

The illustrative decomposition model of table I is simple

[18] An interpretation of the system of table I in ordinary algebraic equations is given in the annex.

Table I

FORMULATION OF DECOMPOSITION MODEL

Resource surpluses			Exogenous	Sector 1				Sector 2				
			Min!	0 ≥	0 ≥	0 ≥	0 ≥	0 ≥	0 ≥	0 ≥	0 ≥	
	Profits →		π_0 =	π_1 =	π_2 =	π_3 =	π_4 =	π_5 =	π_6 =	π_7 =	π_8 =	
Capital: Max!	σ_0	=	350	−1.1	−1.25	−.3	−2.5	−1.0	−2.5	−.6	−3.0	* $y_0(=1)$
Labour	$0 \leqslant \sigma_1$	=	2,000	−12.5	−7.5	−6.0	−7.0	−15.0	−5.0	−4.0	−11.0	* y_1
Sector 1	$0 \leqslant \sigma_2$	=	−50	1	1	−.5	−.2					* y_2
	$0 \leqslant \sigma_3$	=	−50		−.25	1	1					* y_3
Sector 2	$0 \leqslant \sigma_4$	=	−25					1	1	−.8		* y_4
	$0 \leqslant \sigma_5$	=	−25					−.2	−.5	1	1	* y_5
			*	*	*	*	*	*	*	*	*	Shadow prices
			X_0 (=1)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	Activity scales

Feasible basic solutions ("complexes"), sector 1

A	X_1 and X_3	X_1 = 75.000 X_3 = 50.000	L = −1,237.5	K = −97.5
B	X_2 and X_3	X_2 = 85.715 X_3 = 71.429	L = −1,071.4	K = −128.6
C	X_1 and X_4	X_1 = 60.000 X_4 = 50.000	L = −1,100.0	K = −191.0
D	X_2 and X_4	X_2 = 63.158 X_4 = 65.789	L = −934.2	K = −243.4

Feasible basic solutions ("complexes"), sector 2

E	X_5 and X_7	X_5 = 53.571 X_7 = 35.714	L = −946.4	K = −89.3
H	X_6 and X_7	X_6 = 75.000 X_7 = 62.500	L = −625.0	K = −225.0
F	X_5 and X_8	X_5 = 25.000 X_8 = 30.000	L = −705.0	K = −115.0
G	X_6 and X_8	X_6 = 25.000 X_8 = 37.500	L = −787.5	K = −137.5

Fixed costs

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
Capital......	−30	−5	−15	0	−10	0	−10	−5
Labour......	0	−50	0	0	0	−50	0	0

enough to permit a graphical representation by means of an Edgeworth box diagram (figure II). In this diagram the total availabilities of the connecting resources (350 units of capital and 2,000 units of labour) form the edges of the box. Resources used in each sector are measured along the edges in opposite directions. Thus any point in the diagram is a simultaneous representation of four variables: capital and labour used by sector 1, and capital and labour used by sector 2.

Points *A*, *B*, *C*, and *D* in the diagram represent four different *complexes* that can be formed from the activities $X_1 \ldots X_4$ of sector 1; points *E*, *F*, *G*, and *H* represent similar complexes formed from the activities of sector 2. Each of these complexes contains two activities: two is the smallest number that permits the balances of the special resources in each sector to be satisfied.[19] Table 1 contains a list of the activity scales and the total capital and labour requirements of each of these complexes; the respective activity-scale variables are shown near each point in the graph. In figure II, the *efficient* complexes of each sector have been connected by a line. Point *C* represents an inefficient complex in sector 1 since it has larger requirements of *both* capital and labour than point *B*; thus it will never be practical to use complex *C*. Correspondingly, point *G* represents an inefficient complex in sector 2.[20]

The points along a line connecting two complexes (e.g., *A* and *B*) represent weighted averages of these two complexes. For example, the midpoint of the *AB* line represents an average complex which is formed by running projects X_1 and X_3 of complex *A* at half the scales shown in table 1 ($X_1 = 37{\cdot}5$; $X_3 = 25$); likewise running projects X_2 and X_3 of complex *B* at half the scales shown for *B* in table 1 ($X_2 = 42{\cdot}858$, $X_3 = 35{\cdot}715$); and summing the corresponding project scales (only X_3 requires summation; thus $X_1 = 37{\cdot}5$, $X_2 = 67{\cdot}858$, $X_3 = 37{\cdot}715$). It can be verified by simple algebra that the labour and capital inputs of the averaged complex fall exactly halfway between the labour and capital inputs of points *A* and *B*. In the present case, the weighting was 0·5 and 0·5. Points other than the midpoint

[19] These complexes are extreme-point (vertex) solutions of the sub-problems of sectors 1 and 2. These sub-problems are defined algebraically in the annex and are discussed later in the text.

[20] Inefficient points need not use more capital and labour than any *one* point such as *B* or *F*; it is sufficient that they lie northeast (for sector 1) or southwest (for sector 2) of the line connecting such complexes in any sector.

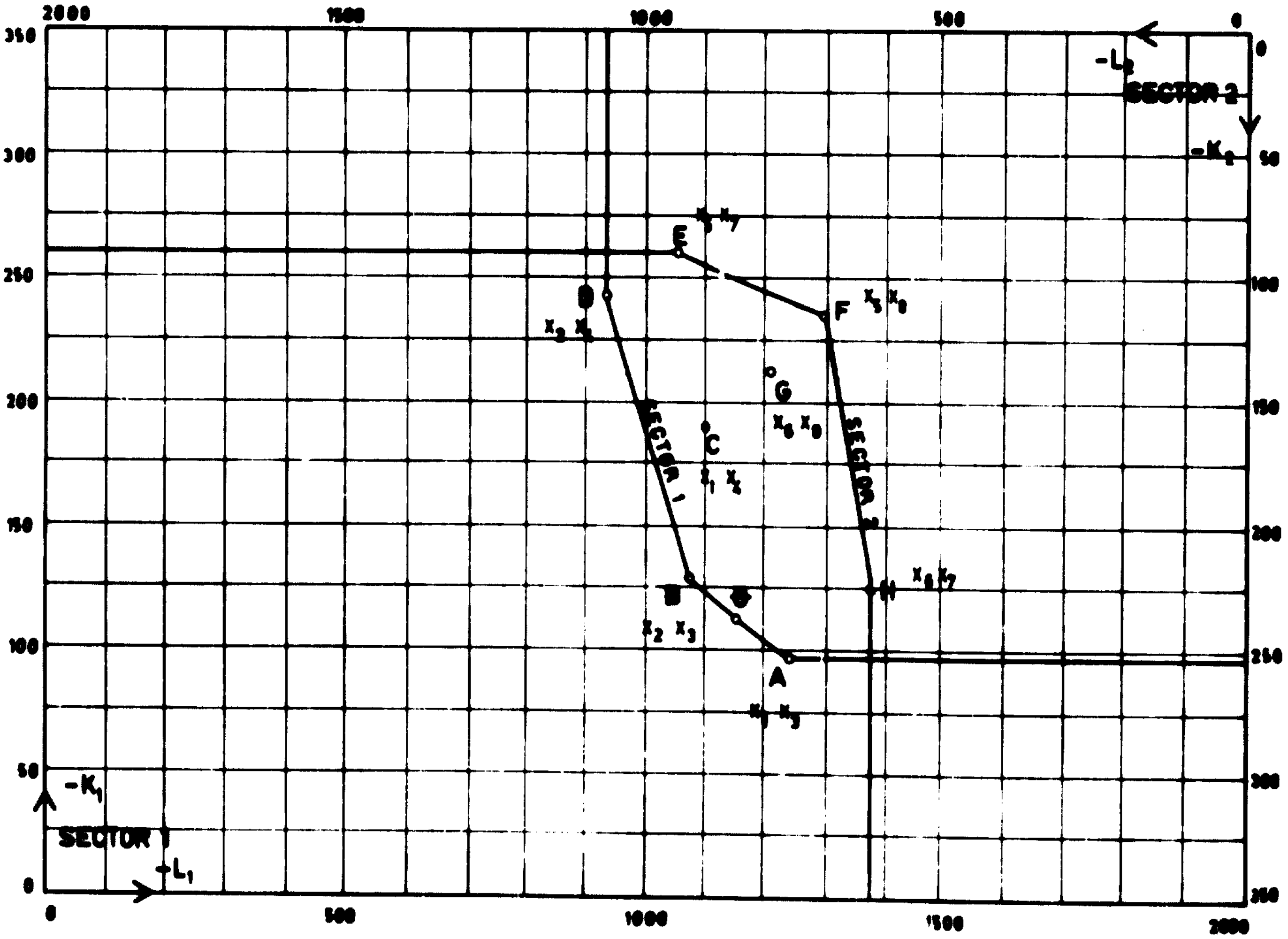

Figure II

LINEAR DECOMPOSITION MODEL

are obtained by using weights in different proportions. The weights may vary from 0 to 1 and have to add up to unity. As long as this weighting rule is observed, the special resource balances of each sector will be satisfied by the averaged complexes, even though the graph contains only the explicit connecting factors. This applies also to any other point that can be attained by starting with the points lying on the connecting line between complexes such as *A* and *B* and then disposing of (wasting, throwing away) some capital and/or labour.

The two curves in figure II can be regarded as generalized iso-product functions for the two sectors describing the alternative combinations of the connecting factors (capital and labour) that can produce the given output of a sector. What is this "given output"? It cannot be identified with any single product since all special sectoral resources are on an equal footing and none can be regarded as *the* product of a sector; it is thus convenient to think of sectoral output as the entire task of satisfying the special resource balances.

The horizontal and vertical extensions of the two sectoral curves to the co-ordinate axes correspond to conventional usage in economies; they signify free disposal of redundant surpluses of the connecting factors.

Figure III provides a graphical illustration of alternative methods of finding an *optimal solution* to the model. Such a solution represents a programme or plan (i.e., a set of activities with determined scales) which is both *feasible*, in the sense that it satisfies all resource balances, and *efficient*, in the sense that it maximizes the surplus of capital (i.e., it minimizes capital requirements).

A *feasible solution* is a programme or plan that satisfies all resource balances but is not necessarily optimal. Points *B* and *T* jointly represent such a plan. Point *B* is on the iso-product line of sector 1; thus it is sure to satisfy the balances of the special sectoral resources in this sector. Point *T* is on the iso-product line of sector 2 and thus satisfies the special resource balances of that sector. The labour requirements of the two points add up to 2,000 units and thus satisfy the labour balance. All of the resource balances are satisfied and the plan is feasible. In order to determine whether it is also optimal, the capital requirements are identified. In figure III they can be seen to leave a capital surplus exactly equal to the vertical distance *BT*. It remains to be decided whether other feasible solutions exist that leave a larger capital surplus.

Note that point *B* is one of the complexes of sector 1 that

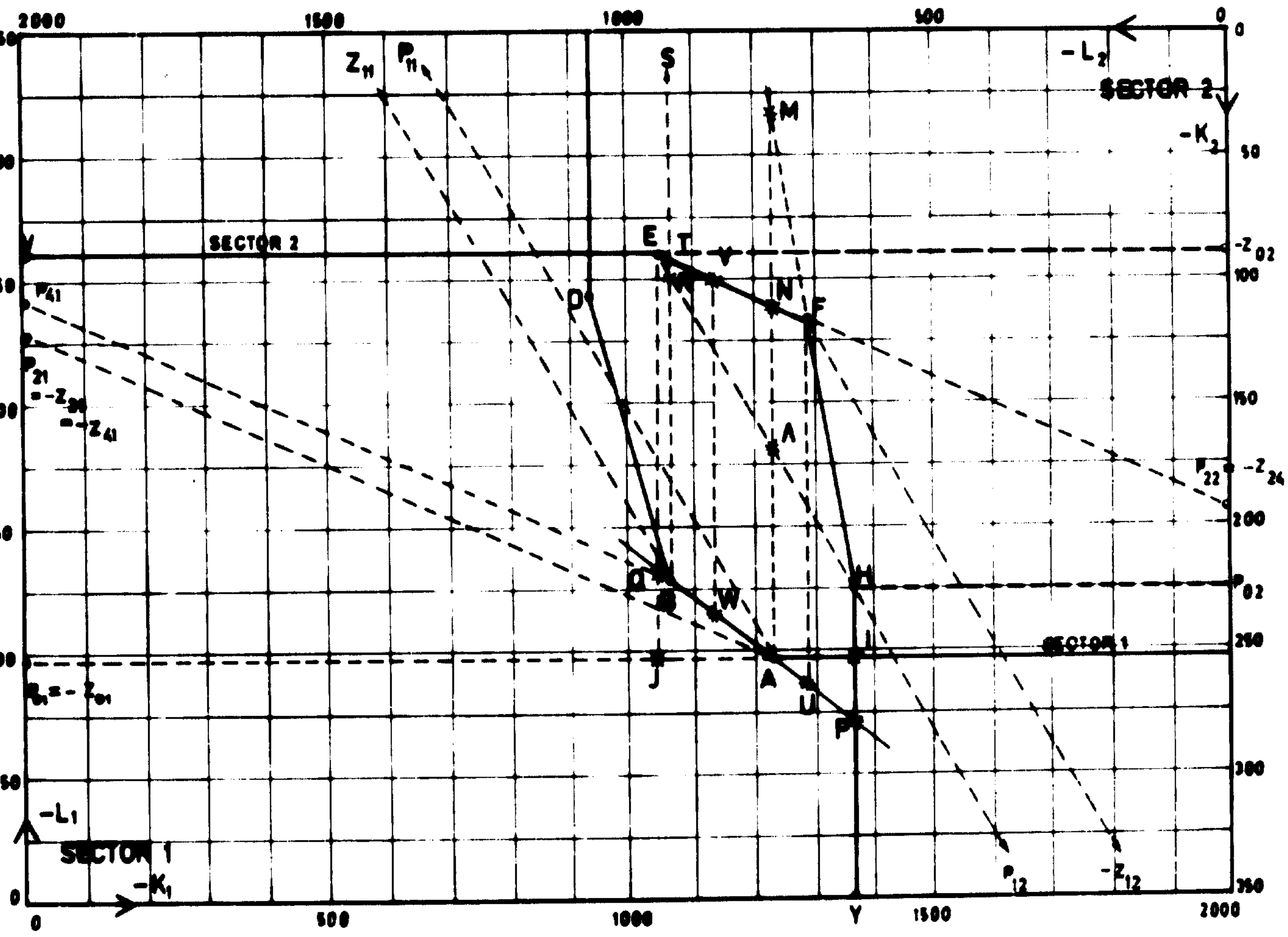

Figure III

Linear decomposition model: graphical solution

has been presented in table 1 while point *T* represents a weighted average of complexes *E* and *F* of sector 2. This solution is labelled as "*BEF*" by reference to the sectoral complexes which form it. Table 2 (line 12) contains a list of the quantitative characteristics of this solution including labour and capital requirements in each sector, capital surplus, and the weights used for averaging in each sector. In sector 2 these weights are 0·926 and 0·074, respectively, for points *E* and *F*; in sector 1, the weight is 1·000 for point *B* since this complex appears by itself, without being averaged with another complex.

In general a feasible solution can be obtained when one point is selected from the iso-product line of each sector, attention being paid to joint labour requirements. When the two points fall on the same vertical line, the joint labour requirements add up to 2,000 units; when the point for sector 1 falls to the left of the point for sector 2, the amount of redundant labour will be equal to the horizontal displacement between the two points (for example, when the combination *AE* is chosen). Conversely, when the point for sector 1 falls to the *right* of the point for sector 2, there will be a labour bottleneck (for example, combination *BE*). As it is generally inefficient to leave labour redundant, a convenient strategy for selecting feasible solutions in the course of optimization is to choose two points that lie on the intersection of a given vertical line with each of the two sectoral iso-product functions. The vertical distance between the two points measures the capital surplus corresponding to the given feasible solution. The geometric determination of the optimum is now obvious: it consists in selecting the vertical line that maximizes the distance between the two sectoral iso-product functions. In the present case the optimum is attained at *AN*; point *N* is a weighted average of complexes *E* and *F* in sector 2. The solution, designated as *AEF*, will be found, quantitatively described, in the sixth line of table 2.

Table 2

LINEAR DECOMPOSITION MODEL: SELECTED SOLUTIONS

Solution	Feasible	λ_{11}[a]	λ_{12}[a]	*Average (1)*	λ_{21}[b]	λ_{22}[b]	*Average (2)*	*Price of labour*[c]	$-L_1$[d]	$-L_2$[d]	K_1[e]	K_2[e]	$\sigma_0 = 350 - K_1 - K_2$[f]
$A\sigma_1H$[g]	yes	1	(137·5)[h]	*I*	1	—	*H*	0	1237·5	625	97·5	225·0	27·5
$A\sigma_1E$[g]	no	1	(183·9)[h]	*J*	1	—	*E*	0	1237·5	946·4	97·5	89·3	163·2
σ_1EH[g]	no	(no point in sector 1: infeasible)											
AEH	yes	1	—	*A*	·428	−·572	*L*	·422	1237·5	762·5	97·5	166·9	85·6
AFH	no	1	—	*A*	1·719	−·719	*M*	1·375	1237·5	762·5	97·5	35·9	216·6
AEF	yes	1	—	*A*	·238	−·762	*N*	·106	1237·5	762·5	97·5	108·9	143·6
EFH	no	(no point in sector 1: infeasible)											
ABH	no	1·828	−·828	*P*	1	—	*H*	·187	1375	625	71·7	225·0	53·3
ABE	no	−·107	1·107	*Q*	1	—	*E*	·187	1053·6	946·4	131·9	89·3	128·8
BEH	yes	1	—	*B*	·945	−·055	*R*	·422	1071·4	928·6	128·6	96·8	124·6
BFH	no	1	—	*B*	3·795	−2·795	*S*	1·375	1071·4	928·6	128·6	192·5	413·9
BEF	yes	1	—	*B*	·926	−·074	*T*	·106	1071·4	928·6	128·6	91·2	130·2
ABF	no	1·346	−0·346	*U*	1	—	*F*	·187	1295·0	705·0	86·7	115·0	148·3

[a] Weights for combining complexes in sector 1.
[b] Weights for combining complexes in sector 2.
[c] Price of labour (price of capital $= P_k = Y_0 = 1$).
[d] Labour requirement (inherently negative) in sectors 1 and 2.
[e] Capital requirement (inherently negative) in sectors 1 and 2.
[f] Surplus of capital (to be maximized).
[g] σ_1 in the solution indicates a surplus of unused labour
[h] The number in parentheses is the value of σ_1.

This geometric method of finding a solution is not applicable to larger problems: Dantzig and Wolfe (1961) however, have provided a generally applicable method which can also be followed by means of the graphical presentation in figure III (see also tables 2 and 3). Dantzig and Wolfe break down the over-all problem into two parts: a "master problem" and "sectoral sub-problems" corresponding to central and sectoral-level planning decisions. The master problem is formulated in terms of the connecting resources, in the present case labour and capital, and is pieced together by averaging *known* sectoral complexes. The master problem represented in figure III, also determines prices for the connecting resources; in the present case, a price ratio for labour and capital. The sectoral sub-problems, on the other hand, systematically find previously unknown sectoral complexes for inclusion in the master problem. The sectoral sub-problems do not appear explicitly in the graph of figure III, but compliance with their balances is guaranteed by the averaging rules discussed above. The starting point of the technique has to be one known basic feasible solution to the master problem; given such a starting point,[21] the interaction of the two parts of the problem guarantees the attainment of the optimal solution in a finite number of steps.

A *basic solution* contains the smallest number of non-zero variables that is compatible with the number of equations. In the master problem we have four equations (see annex): one for balancing capital and labour requirements respectively, plus one in each sector for describing the averaging

[21] If no basic feasible solution is known that would be suitable as a starting point, it is possible to construct one by algebraic techniques (Dantzig and Wolfe (1961)).

Table 3

LINEAR DECOMPOSITION MODEL: SOLUTION PATHS

Number of Solution	Solution	p_1 [a]	$-z_1$ [b]	$p_1 + z_1$ [c]	p_2	$-z_2$	$p_2 + z_2$	Variable in	Selection	Variable out	Average Solution	Feasible
0	$A\sigma_1 H$	97.5	97.5	0				–				
					225.0	89.3	135.7	*E*	√	$A\sigma_1 HE$	–	
										$A\sigma_1 HE$	[illegible]	√
										$A\sigma_1 HE$	[illegible]	
1	*AEH*	619.7	580.7	39.0				*B*				
					488.8	412.5	76.3	*F*	√	*AEHF*	–	
										AEHF	*M*	
										AEHF	*N*	√
2	*AEH*	228.7	228.7	0				–	OPT			
					189.6	189.6	0	–				
		Alternate path if complex "B" is chosen as incoming variable in solution 1 above										
1	*AEH*	(As above)						*B*	√	*AEHB*	*R*	√
										AEHB	*P*	
										AEHB	*Q*	
3	*BEH*	580.7	580.7	0				–				
					488.8	412.5	76.3	*F*	√	*BEHF*	–	
										BEHF	*S*	
										BEHF	*T*	√
4	*BEF*				189.6	189.6	0	–				
		242.2	228.7	13.5				*A*	√	*BEFA*	*V*	√
										BEFA	*U*	
										BEFA	*Q*	
5	*AEF*	(As above)							OPT			

a "Subcontracting fee"–a revenue.
b Optimal combined factor cost (z · 0).
c Profit on optimal complex at current prices.

rules for complexes. There are two kinds of variables in the master problem: the weights to be applied to the individual complexes of each sector, and capital and labour surpluses that can also be interpreted as disposal activities. Generally, at least four[22] of these variables must be non-zero. One will be the capital surplus σ_0 which is being maximized, and the other three may be three sectoral complexes, or two complexes and the labour surplus (disposal) activity σ_1. In figure III, basic feasible solutions are obtained, as before, by selecting intersection points of a vertical line with the iso-product curves, but with the additional restriction that the vertical line has to run through a vertex (a point for a single complex) in *one* of the sectors.[23] Solutions *BEF* and *AEF* are such basic solutions, but solution *ABEF* – corresponding to the vertical line *VW* – is not, as it contains five non-zero variables: capital surplus (the maximand), plus non-zero weights for each of the four complexes *A* and *B* in sector 1, and *E* and *F* in sector 2. A solution such as *AEH*, corresponding to the vertical line *A*, is also a basic feasible solution, even though it is off the iso-product line of sector 2, since the point Λ can be obtained by averaging the two non-neighbouring complexes *E* and *H*. This point is, of course, not efficient since it could also be attained by starting with point *N* on the iso-product curve and then wasting some capital (corresponding to the distance *N*Λ).[24]

In the master problem, not only the starting solution, but all later solutions also have to be basic because only basic solutions determine a unique price ratio for labour and capital, a ratio which is needed in the sectoral sub-problems.

[22] The number of variables including slacks (surpluses) in a linear programming problem exceeds the number of equations; the difference is known as the number of *degrees of freedom* of the system. A corresponding number of variables can be fixed arbitrarily, and the values of the remaining variables are then determined by solving the system of simultaneous equations. If the preset variables are assigned the value of zero, we get a basic solution. By coincidence, the solution value of one or more of the variables that have *not* been pre-set may also turn out to be zero; in this case the number of *non-zero* variables will be less than the number of equations. Such a solution is termed "degenerate".

[23] Degenerate solutions are obtained when, *by coincidence*, complexes in both sectors fall on the same vertical line.

[24] Basic solutions need not be feasible. If the solution value of any variable (a weight or a slack) turns out to be negative, the solution is infeasible. In figure IV, basic but infeasible solutions are obtained if the vertical line is made to intersect not the line segment connecting two vertices but the continuation of such a line segment beyond one of the vertices. This represents an impermissible weighting of the two complexes with one weight negative and the other exceeding unity. See for example point *P* corresponding to the averaging of complexes *A* and *B* in solution *ABH* (table 2, line 8).

In a basic solution the price ratio is fixed by the slope of the averaging line segment that is intersected in one of the two sectors. If the solution is non-basic, such as *ABEF*, the vertical line $I'H'$ intersects line segments, generally of *different* slopes, in both sectors rather than passing through a vertex in one sector.

Let us now trace the course of optimization, using the Dantzig-Wolfe algorithm, by reference to figure III. Suppose the starting point is at the vertical line *HI*. This corresponds to a basic feasible solution (labelled "$A\sigma_1 H$" in table 3) in which complex *A* in sector 1 and complex *H* in sector 2 appear with unit weights; thus two weighting variables are non-zero. In addition, there is some labour disposal and thus the labour surplus variable σ_1 will also be non-zero; its value corresponds to the distance *AI*, which amounts to 13·5 units. The value of the maximand (the capital surplus variable σ_0) corresponds to the distance *AI*, or 27·5 units.

We assume that at this point only complexes A *and* H *are known.* While there are only six efficient complexes in this problem, in larger problems the number of possible complexes increases in combination and thus *at the beginning of the optimization there exists very little information concerning alternative efficient sectoral complexes.* The task of the sectoral sub-problems is to identify precisely previously unknown efficient sectoral complexes for inclusion in the master problem.

Looking at it another way, if all the efficient sectoral complexes were known from the very beginning, the optimal solution to the master problem would immediately give the optimal solution to the problem as a whole. However, as we are generally working with an incomplete list of complexes, we require a technique that will bring to light new complexes; specifically, we have to discover those complexes that are needed for the optimal solution of the over-all problem without having to enumerate all possible efficient sectoral complexes. We shall now indicate how the sectoral sub-problems are utilized to achieve this aim.

In the starting solution the price ratio between labour and capital is determined by the slope of the line segment *AI*, i.e., the price of labour is zero. The price of capital is unity by assumption. Using these relative prices, the sectoral sub-problems *maximize the combined value of the connecting resources.* In the present problem the connecting resources appear as inputs; thus we are, in effect, minimizing their combined cost. At the same time, the sectoral sub-problems must satisfy the balances of the special sectoral resources.

Although the special resource balances of the sectors are not explicitly shown in figure III, they are nevertheless allowed for by means of the averaging rules applicable to complexes. The straight lines connecting the points corresponding to the sectoral complexes represent weighted averages of complexes; as long as the complexes themselves satisfy the special sectoral resource balances, these weighted averages will also satisfy the special balances. Moreover, when we take one of the points corresponding to the complexes or their weighted averages and subsequently dispose of (throw away) some labour or capital, we are still certain to satisfy the same sectoral balances. Thus we can map out *a feasible area* for both sectors in the graph. These areas consist of the iso-product lines *plus* all the points falling on the concave sides of these lines. Whenever a point is chosen within the feasible area of a given sector, the special sectoral resource balances are certain to be satisfied. In this way we can use the graph of the master problem to represent possible solutions to the sectoral problems.

The question arises if in maximizing the combined value (minimizing the cost) of the connecting resources in the sub-problems, using the price ratio of the starting solution, we discover new complexes that are more efficient than the ones already known. In figure III, the combined value of the connecting resources is represented by *budget lines* whose slope equals the price ratio between labour and capital and whose intercept on the capital axis measures this combined value.[25] The optimization in each sector is represented by a parallel shift of the budget line in such a way that the combined value of connecting resources is increased (combined cost is decreased) while maintaining at least one point of the budget line within the feasible area of the sector. In sector 1 this procedure leads to point *A*, which had already been known previously, but in sector 2 the optimum corresponds to a new complex *E* whose exact capital and labour requirements are disclosed by the optimization process.

In what sense is complex *E* more efficient than previously known complexes? In the starting solution (figure III), *H* was the only known complex for sector 2. The combined cost of the connecting resources for this complex can be read off by tracing a budget line with slope 0 to the capital axis of sector 2: in figure III we read off 225 units at p_{02} (the same value will also be found in table 3, in the line of solution 0 labelled "$A\sigma_1 H$", under p_2).[26] The combined

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[25] The budget line corresponds to the equation

$$P_L.(-L) + P_K.(-K) = (-z),$$

or:

$$(-K) = (-Z) - P_L.(-L),$$

since $P_K = 1$. On the graph the axes correspond to $(-K)$ and $(-L)$; thus $(-z)$ is the intercept on the $(-K)$ axis.

[26] In the master problem p_2 is a shadow price that corresponds to the equation describing the averaging rule for sector 2 (see annex). Whenever a complex is included in a basic solution, i.e., when its weight is non-zero, the shadow profit for the column of this complex has to vanish. The mathematical reason for this is the well-known rule of *complementary slacks* applicable to linear programming problems; in economic terms the solution enforces "perfect competition" between all complexes included in it. Consequently, the shadow price p_2 and the combined value of the connecting resources have to add up to zero; i.e., the combined value equals $-p_2$.

The above p_2 can conveniently be interpreted as a "subcontracting fee". The master problem places all complexes of a sector in competition with each other for the privilege of performing the task of the sector, namely satisfying the balances of the special sectoral resources. Whichever complex or complexes can perform this task at the lowest subcontracting fee will be selected to do the job. At any stage, the successful complexes will just break even; their combined cost for the connecting resources at the prevailing prices will just equal the subcontracting fee. The solution to the master cost for complex E is, however, only slightly under 90 units as read off in the graph at $-z_{02}$ (89·3 units under $-z_2$ in table 3). Consequently, the inclusion of complex E in the solution promises a combined cost improvement of 22·50 − 89·3 = 135·7 units, at the prevailing prices.

In order to advance from the starting solution we will want to include E in the next solution of the master problem. As the solution is to be basic, we will have to drop some other complex or the labour surplus (disposal) activity. Table 3 indicates the three ways of dropping variables and the corresponding solutions; the capital surplus activity which is to be optimized is never dropped. If we drop complex A, we are left with no complex in sector 1, and thus we have an infeasibility. If we drop σ_1, we get solution AEH which yields an average complex for sector 2 at point A, a feasible solution. If we drop complex H, we get solution $A\sigma_1E$ which leads to point J for sector 1, an infeasible point, implying a negative σ_1. (Numerical data describing each of these trial solutions will be found in table 2.) Thus we have only one feasible choice: solution AEH. This is labelled as solution 1 in table 3.

AEH determines a price ratio of 0.422 between labour and capital: this ratio equals the slope of the line connecting E and H. Budget lines with this slope yield new complexes in the course of the optimization in both sectoral subproblems: in sector 1, the new complex is B, with a combined cost of connecting resources equal to $(-z_{11})$ = 580·7; in sector 2 the new complex is F, with a combined cost of $(-z_{12})$ = 488·8. The cost improvement relative to solution AEH can be determined by comparison with the combined cost of A in sector 1 which equals 580·7 (p_{11} in figure III; also in table 3), and the combined cost of either E or H (these are equal) in sector 2 which equals 412·5 (p_{12} in figure III; also in table 3). The cost improvements are thus 39·0 and 76·3 units in sectors 1 and 2, respectively.

Either of these new complexes can be included in the solution of the master problem to obtain an improvement in the maximand σ_0; it is preferable, however, to include the one with the larger cost improvement, namely F. Once again a variable must be dropped in order to keep the solution basic; the three choices are indicated in the line of solution 1 in table 3, and the resulting alternative solutions are numerically specified in table 2. The only feasible choice is AEF, which determines a price ratio of 0·106 (equal to the slope of the segment EF). At this price ratio the budget lines disclose no new complexes in the course of the sectoral optimizations, and thus the solution AEF turns out to be optimal.

If, at the stage of solution 1, complex B had been included in the next solution rather than complex F, the path of optimization would have been slightly longer. In this case BEH turns out to be the next feasible solution; the price ratio remains 0·422 as in solution 1. At this price ratio, F is still present with a potential improvement and is thus the next complex to be included in the master solution. The next feasible solution is obtained by dropping H; thus solution 4 is BEF, with a price ratio of 0·106. At this price ratio point A appears as an improved point in sector 1; the next feasible solution, after dropping B, is AEF, the optimal solution.

From the point of view of decentralization this analysis of the decomposition algorithm is significant in that it discloses the insufficiency of price-type control instruments in attaining an optimal solution. As already discussed by Clopper Almon[27] the upper decision-making level cannot guarantee the balance of connecting resources merely by setting the prices of these resources, in a solution such as AEF the price ratio EF will *not guarantee* that sector 1 will choose to produce exactly with the weighted average N of complexes E and F. Faced with the price ratio EF this sector may produce at any point along the segment EF, as all points along this segment are equally optimal at the stated price ratio; it makes no difference which point is taken when dealing with sector 2 alone. If the central planning office wants to ensure an adequate balance of the connecting resources, it has to prescribe a weighting of complexes E and F in sector 2, or a quantitative allocation of labour and capital to this sector. At the same time, sector 1 can be adequately regulated by the price ratio alone, since at the given price ratio it has a unique equilibrium position at A.

An interesting feature of the practical application of control instruments in this situation is that the upper decision-making level will find it worth while to use *both* price and quantity-type control instruments, even though their joint use will be redundant in sector 2.

> "They (the Central Trade Office) announce in quantitative terms their feasible plan. They tell each plant manager how much of each traded commodity he must produce and how much he is allowed to purchase . . . They also announce the prices and direct that trade be conducted at these prices. *They may also instruct the managers that, subject to their meeting the quantitative goals . . . they should also maximize profits. Such a rule is intended as a guide to avoid possible waste in the event that* S *(the quantitative goal) is not precisely achieved for one reason or another.* It is important to note that they cannot tell the managers simply to maximize profits (omitting production goals, S) for if they did, Central Trade would almost certainly have difficulty with its constraints."[28]

At the level of activities within a sector, e.g., the project level, this insufficiency of price-type control instruments is translated into the insufficiency of the usual price-type project evaluation criteria, and calls attention to the fact that *there is an inescapable minimum of quantitative control that*

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problem can be improved, however, as long as sectoral optimization discloses new complexes that can make a profit at the prevailing process and prevailing subcontracting fees. When this is no longer possible, an over-all optimum for the entire problem is attained.

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[27] In Dantzig (1963), pages 462–465.

[28] Almon in Dantzig (1963), pages 464–465 (emphasis added).

Yarn being wound onto ring bobbins in a cotton-textile mill at Khartoum, Sudan

A textile plant at Caricuao, Venezuela *as to be exercised even in highly decentralized linear systems.*[29] This does not mean that multi-level planning is useless; on the contrary, it reinforces the need for such planning as it indicates that a decentralized market mechanism without a central decision-making level will encounter the same indeterminacies that characterize the multi-level planning system with pure price-type co-ordination. At the same time multi-level planning is preferable to pure central planning, as it results in an economy of information flow. It should be noted that the master problem in the decomposition algorithm requires no information on special sectoral resources or on particular sectoral projects or activities; this information is dealt with indirectly by delineating feasible regions for each sector on the basis of averaging known sectoral complexes.

The decomposition algorithm of Dantzig and Wolfe is not the only one that can be utilized for co-ordinating the master programme with the sectoral sub-programmes. Kornai and Liptak (1965) have proposed a multi-level planning system in which the information flow is the reverse of that in a Dantzig-Wolfe system. In a Dantzig-Wolfe decomposition the master programme signals prices to the sectoral sub-problems and the latter signal combined utilizations of interconnecting resources by particular complexes to the master programme; in other words, prices flow downward and quantities flow upward (except for the quantitative implementing objectives fixed by the master programme for the sectors in which averaging is required). In the Kornai-Liptak decomposition the master programme passes allocations of the connecting resources to the individual sectors; the sectors, in turn, signal their own sectoral shadow prices for these resources to the master programme. Without going into the details of the Kornai-Liptak decomposition it can be seen (figure III) that sectoral resource allocations of labour can be represented by a vertical line cutting the two iso-product curves; at any (basic or non-basic) solution separate shadow prices can be determined for each sector. For an averaged complex, the shadow price coincides with the slope of the averaging segment; for a single complex (which appears with unit weight) the shadow price is distinct for increased and for decreased allocations. For *non-optimal* solutions the comparison of shadow prices for the two sectors will show an unambiguous difference; for example, for the basic solution *BEF* the shadow price of labour, both in the upward and the downward direction, is greater in sector 1 than in sector 2. This indicates the need for increased labour allocation to sector 1 at the expense of sector 2. Conversely, for the basic solution $A\sigma_1 H$, an unambiguous price difference exists in the opposite sense, indicating the need for increased labour allocation to sector 1 at the expense of sector 2. *At the optimum* (solution *AEF*) the vertical cut through *A* and *N* will yield a shadow price at *N* that is smaller than the shadow price at *A* for decreased labour allocation to sector 1, and larger than the shadow price at *A* for increased labour allocation to sector 1, thus indicating a stable equilibrium.

The decomposition principle in non-convex systems

We shall now use the diagrammatic method developed for linear decompositions to indicate the changes that are introduced by non-convexities, as represented by the case of fixed costs. The principal change concerns the applicability of iterative corrections to such systems in order to improve the efficiency of existing feasible solutions, as these tend to break down in the presence of non-convexities. One has the intuitive feeling that the presence of small non-convexities cannot have a profoundly disturbing influence on the behaviour of largely convex systems, as common observation indicates that markets are often able to operate with reasonable efficiency despite the pervasive presence of fixed costs, economies of scale, and other non-convexities. But what is "small"? What systems are "largely convex"? The diagrammatic method offers some bases for judgement on these points and suggests guidelines for workable if not perfect decentralization.

Figure IV indicates the first step in constructing a decomposition diagram with fixed costs included to represent non-convexities. The fixed costs are expressed in terms of labour and capital requirements (see table 1). For each complex such as *A*, *B*, etc. the fixed costs of the component projects (activities) are added up. In figure IV, these additions are performed by means of *vectors* (arrows) which represent the labour and capital requirements of individual activities. In this fashion, point *A* is carried into point *A'*, point *B* into point *B'*, and so on. While points *A*, *B*, . . . in the diagrams have been referred to as vertices, we shall refer to points *A'*, *B'*, . . . as *apices* in order to distinguish clearly between the two points.

Generally, apices cannot be averaged in a linear fashion, because averaging apex *A'* and *B'*, for example, requires the joint use of projects X_1, X_2, and X_3, while apex *A'* allows only for the fixed cost of X_1 and X_3, and apex *B'* only for X_2 and X_3. Thus when two complexes are to be used jointly and all the fixed costs have been incurred, the variable costs can be averaged linearly.[30] In figure V these operations have been performed; for example, at *A'* the vector $\bar{x}_2$ has been added, while at *B'* the vector $\bar{x}_1$ has been added; the end-points of these two vectors can now be connected by a straight line. It is significant that the slope of this *correct averaging line* for apices *A'* and *B'* is the same as

[29] The range of indifference in the solutions of some sectors at stated connecting-resource prices is eliminated in *strictly convex* systems which may *not* have linear boundary segments. Only in such systems will prices alone suffice to achieve decentralization.

[30] If fixed costs also comprise requirements of special sectoral resources these requirements can be translated into equivalent labour and capital requirements calculated at the marginal labour and capital requirements needed for producing the specified amounts of sectoral resources, on the assumption that all of these sectoral resources will in fact be produced in the optimal programmes and that the corresponding fixed costs will thus be incurred in any event. This assumption may not be valid; and there might be some choice in the selection of activities for producing these fixed-cost components. We shall abstract from all of these secondary complications in the course of the present discussion.

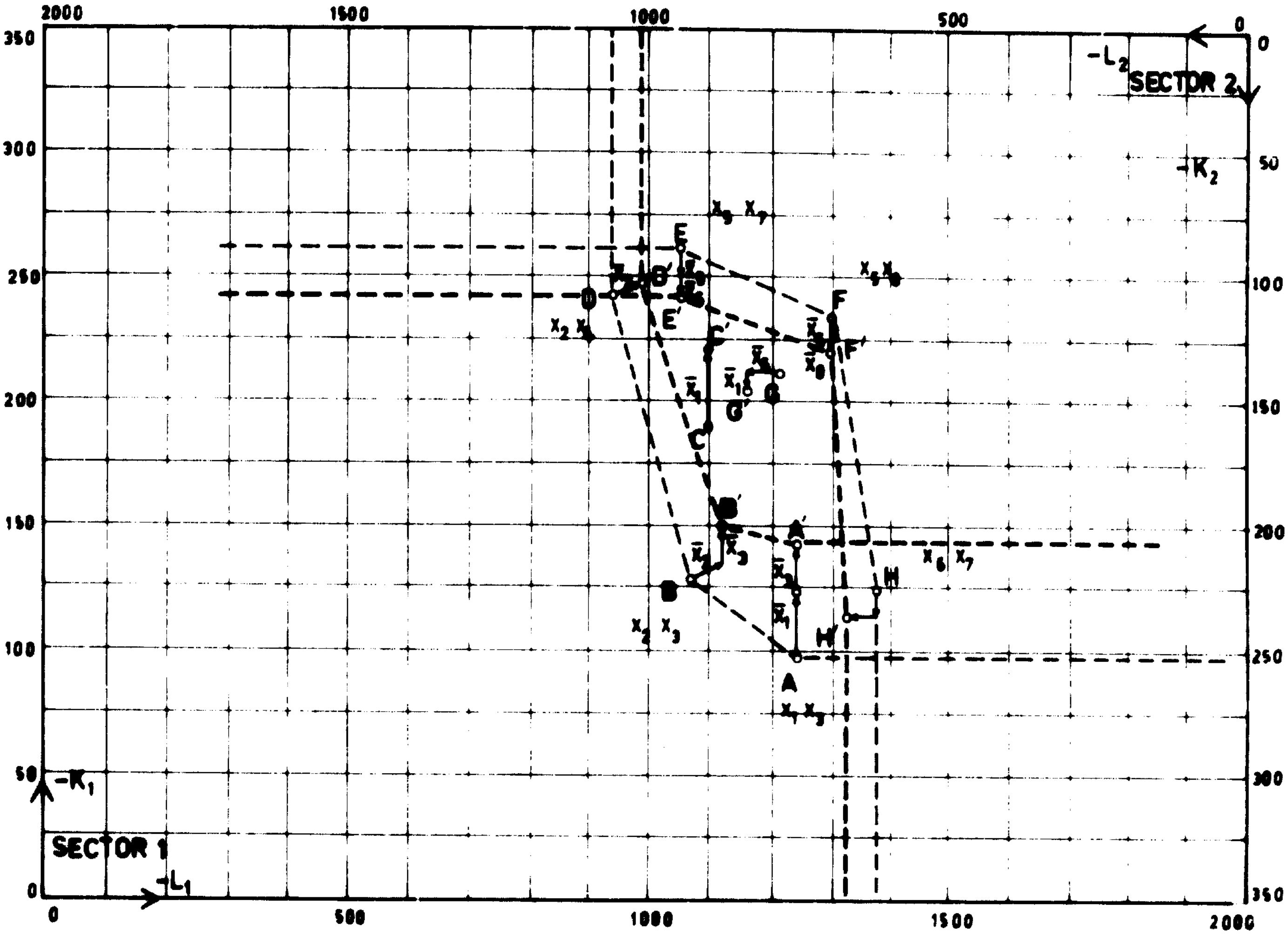

Figure IV

DECOMPOSITION: FIXED COSTS OF COMPLEXES ADDED

the slope of the vertex-to-vertex average. This is due to the fact that *A*, *B*, and the end-points of the correct averaging line form a parallelogram, because *the same three vectors* have been added both to *A*, and to *B*, though in a different sequence. Thus the correct averaging line reflects *marginal* rates of substitution between labour and capital, while an *apex-to-apex connecting line* does not.

Two important qualifications to the foregoing procedure have to be noted:

(*a*) While points *C* and *G* represent inefficient complexes in a linear system, it is by no means a foregone conclusion that they will also be inefficient in a non-convex system comprising fixed costs. If, for example, the fixed costs associated with *C* were unusually small, it could easily happen that the correct averaging line involving *C* will pass in part on the *infeasible* side of the correct averaging lines for the other complexes, and will thus yield preferred points in this range (see figure VIII and footnote 30).

(*b*) In a linear system, averages of neighbouring vertices are always superior to averages of non-neighbouring vertices. In a non-convex system with fixed costs this is not necessarily so; for example, the correct averages between Apex *A′* and *B′* and between apex *B′* and *D′* may prove inferior in certain ranges to the correct average of apex *A′* and *D′* if the fixed costs associated with vertex *B* are unusually high.

Do the apices and the correct averaging lines appearing in figure V jointly form iso-product lines for the two sectors? In answering this question it should be noted that free labour and capital disposal is permitted at all times; thus any point in the diagram representing a legitimate apex or average will dominate all points derived from it by such disposal activities. Therefore *B′* will dominate all points on the correct averaging line between *A′* and *B′* that are to the northeast of *B′*; and likewise for *A′*. As a result, the entire line connecting the end-points of vectors $\bar{x}_1$ added to *B′* and $\bar{x}_2$ added to *A′* will disappear and will be replaced by a step function between *A′* and *B′* (figure VI). Applying the same considerations of dominance to other areas of the diagram, we obtain the iso-product lines of figure VI which have a much simpler configuration than the apices and correct averaging lines of figure V. This simplification of the diagram is not a special feature of the numerical example under study but a general phenomenon which is due

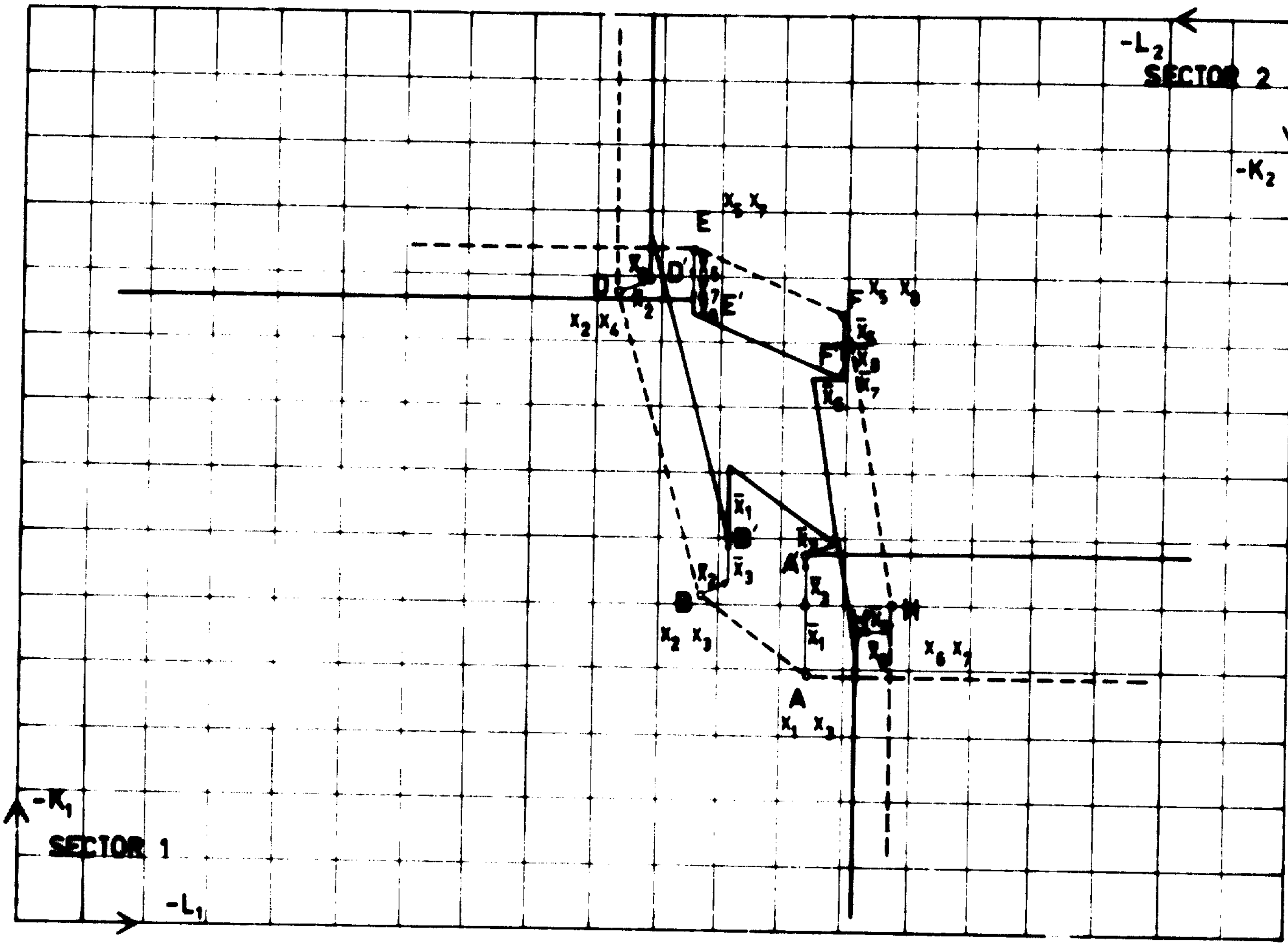

Figure V

DECOMPOSITION: AVERAGING OF COMPLEXES INTRODUCED

to the fact that the correct averaging lines have pronounced dips at the apices where *one* fixed cost is in all cases eliminated. As a result the straight line segments representing variable costs are generally truncated near the apices and in some cases (as between *A'* and *B'*) completely eliminated in favour of simple step functions.

What can be said about the non-convex decomposition problem represented by the iso-product lines of figure VI? In general when the lines are correctly drawn and all the apices corresponding to feasible basic solutions of the sectoral problems are known, it is impossible to find a solution to the master problem without taking into consideration all the detailed information represented by the specific sectoral resource balances and sectoral activities. A knowledge of the capital and labour requirements at these apices, together with correct averaging procedures, is sufficient to guarantee an exact solution to the master problem. The averaging procedure in the present case can be based on a listing of activities included in each complex together with their fixed capital and labour requirements; when two or more complexes are averaged, it is then necessary merely to check off all activities that are included and to add up their fixed costs. Formally, the master problem becomes an integer programming problem in which the averaging of the variable costs of the complexes is conditional on incurring all the requisite fixed costs (see annex).

In practical applications, the shortcoming of this procedure is twofold. First, it is difficult to solve a large integer programming master problem; second and more important, the availability of information concerning the requisite apices cannot be taken for granted, because the number of such apices increases in combination with the size of the problem. The virtue of the Dantzig-Wolfe algorithm is precisely that it generates new complexes as they are needed, thereby shortcutting the enumeration of efficient complexes. The question is, can a similar procedure be developed for the non-convex case? No such procedure is presently available and the difficulties of evolving one are great.

First, the meaning of prices in the master problem — as in integer programming problems in general — now becomes ambiguous. In figure VII the over-all optimum happens to be at the vertical line passing through *B'*, as can be verified geometrically or by means of a simple enumeration of

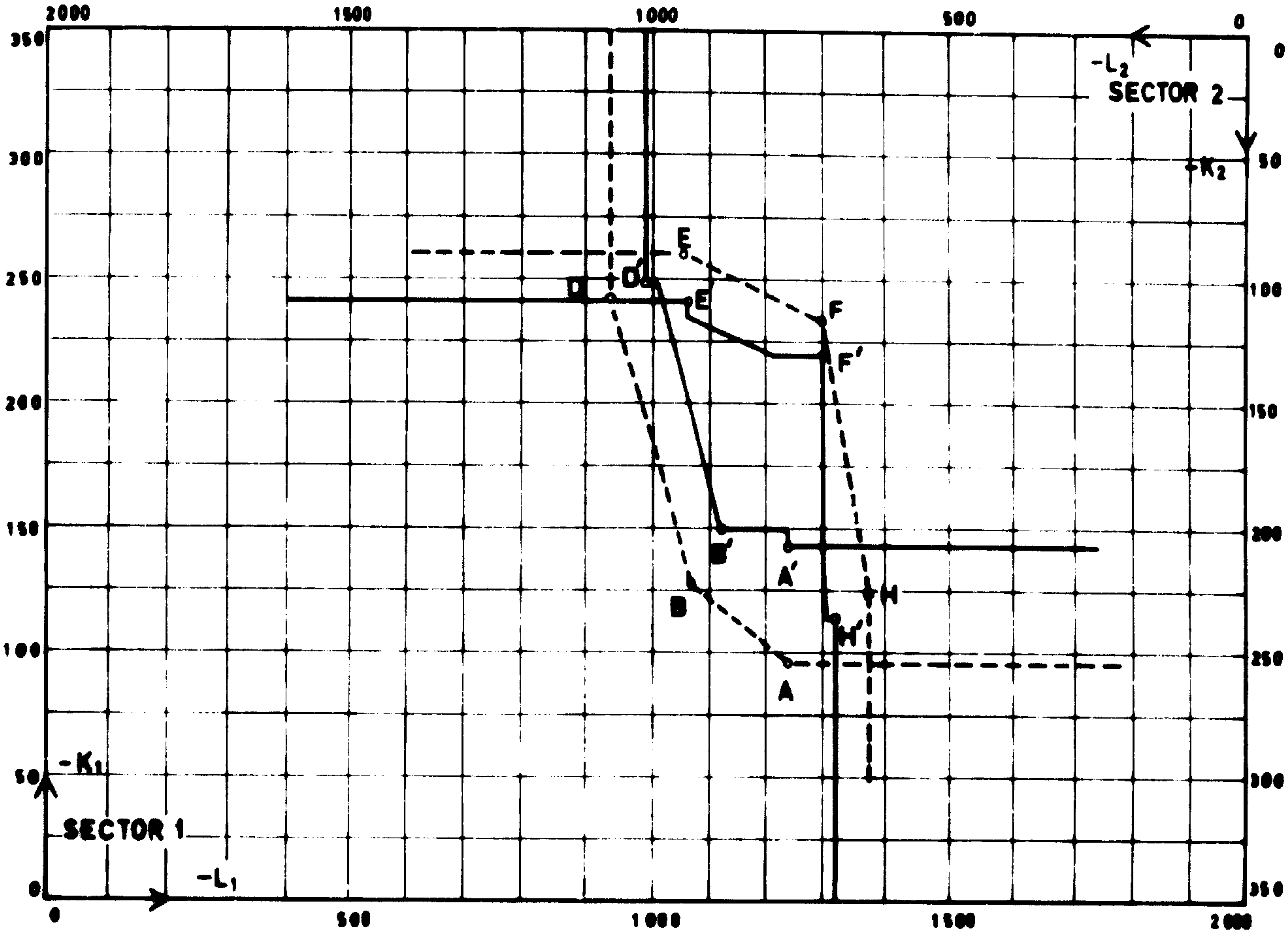

Figure VI

DECOMPOSITION: ISO-PRODUCT LINES DERIVED

alternatives. What is the proper price ratio between labour and capital characterizing this optimum? Is it the slope of the iso-product line at *J*? This slope corresponds to the averaging of variable costs, i.e., to the slope of the line *EF*; it is thus a marginal cost ratio. Or is the proper price ratio the slope of the apex-to-apex connecting line, *E'F'*? In the present case the two slopes are not greatly different, but with only a small change in some of the fixed costs the optimum can be shifted to a vertical line such as *MN*. Here we have three possible price ratios: the two above-mentioned, and the zero price corresponding to labour disposal.

Second, we have to ask what the role of such a price ratio is going to be. Will it be used, as in the linear decomposition problem, in a search for new efficient complexes? If so the sectoral sub-problems become integer programming problems involving the minimization of combined costs (as in the linear case), but with allowance for fixed costs of the individual projects. In the present illustrative case (figure VII) such sectoral optimizations performed at the proper price ratios will identify all apices that participate in defining the iso-product lines; however, this cannot be generally guaranteed, because apices can also occur in local indentations of the iso-product lines that are not optimal under any price ratio. In figure VIII, for example, the fixed costs of sector 1 have been changed and apex *C'* now occurs within a local indentation of the iso-product line.[31] Regardless of the price ratio for the optimization performed within sector 1, apex *C'* will never become the optimum. If the price ratio between labour and capital is

[31] Fixed costs within sector 1 have been changed as follows: $\bar{x}_1 = 0, 0$; $\bar{x}_2 = 0, 0$; $\bar{x}_3 =$ 100, 100; $\bar{x}_4$ 20, 20. These fixed costs carry the points *A*, *B*, *C*, and *D* into *A'*, *B'*, *C'*, and *D'*, as shown in figure VIII. The correct averaging lines for all binary combinations of complexes are shown: these are denoted by *ab*, *ac*, *ad*, *bc*, *bd*, and *cd*. The sectoral iso-product function is made up of a vertical stretch above *D'* (capital disposal together with use of complex *D'*); line *cd* (the correct averaging line for complexes *C'* and *D'*); the line segment between points *C'* and *U* (use of complex *C'* together with labour disposal); the line segment between points *U* and *V* (part of the correct averaging line *bd*); the line segment between points *V* and *B'* (use of complex *B'* together with capital disposal); the line *ab* (correct averaging line for complexes *A'* and *B'*); and the horizontal stretch to the right of *A'* (use of complex *A'* together with labour disposal). The correct averaging lines *ac*, *ad*, and *bc* lie entirely above the iso-product functions, as do parts of *bd*. The line connecting points *B'* and *D'* is *not* a correct averaging line.

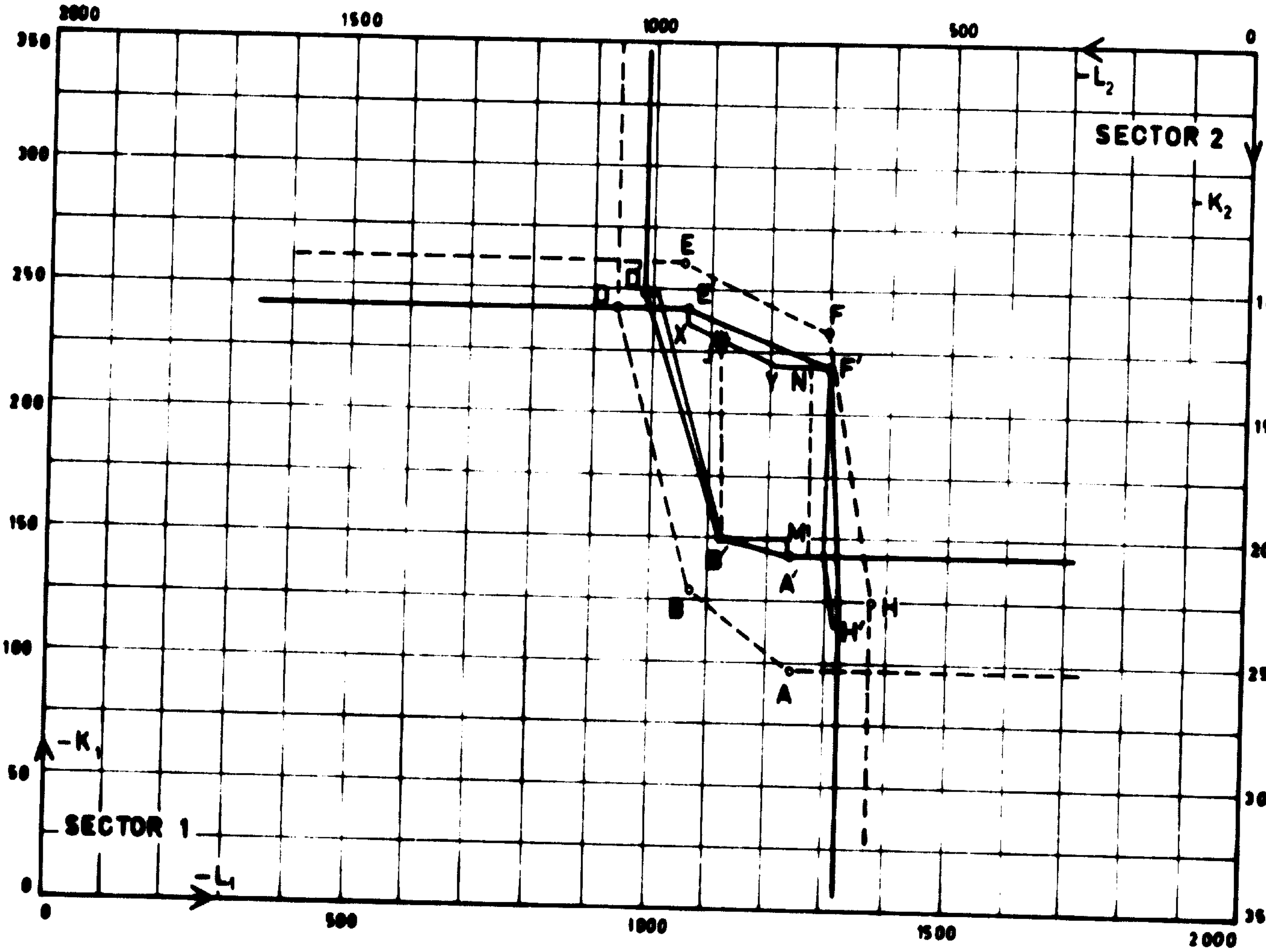

Figure VII

DECOMPOSITION: APEX TO APEX CONNECTING LINES COMPARED TO ISO-PRODUCT LINES

higher than the slope of the line $B'D'$, apex D' will be optimal, but if it is lower, apex B' will be optimal. Thus the sectoral optimization can identify apices only if they lie in the convex hull of the sectoral iso-product lines.

Alternately, the role of the price ratio may be to sustain an optimum, as in the linear case; if so, the local marginal price is the proper one to use, but under the assumption of profit maximization for each sector, such a price will sustain the optimum only in a most unstable way as the slightest change in the price ratio will generally precipitate a cumulative movement away from the optimum. The concept of price characterizing convex systems is obviously not capable of ready superficial extension to non-convex systems.

Although no available procedure *guarantees* the iterative derivation of the *exact* optimum while shortcutting the enumeration of efficient complexes, we may still make considerable headway toward the practical objectives set out at the beginning of this section by looking for suitable approximations.

Figures VII and IX have been drawn to indicate two possible approximations to the derivation of an exact optimum in such non-convex decomposition problems.

Approximation 1 (apex-to-apex connecting line)

In figure VII the apex-to-apex connecting lines are shown in relation to the correct iso-product lines. *The apex-to-apex connecting lines yield a linear approximation to the non-convex master problem* while maintaining the non-convex nature of the sectoral problems. The linearized master problem in effect assumes *perfect divisibility* of the sectoral complexes, and thus ignores the all-or-nothing character of fixed cost incurrence in a given sectoral activity. An approximate over-all solution can be obtained in an iterative fashion by determining successive price ratios from the basic solutions of the linearized master problem; these price ratios are then applied to sectoral *integer* programming problems in an attempt to identify new efficient apices (sectoral complexes), if such are available. These new apices if found, are included in the linearized master problem and the procedure is iterated. This procedure has the virtue of generating new complexes only as needed, similarly to the linear decomposition problem (see annex). The key characteristics of approximation are:

(*a*) It will always yield either an exact or an *over*estimate of the correct optimal value of the objective function. The

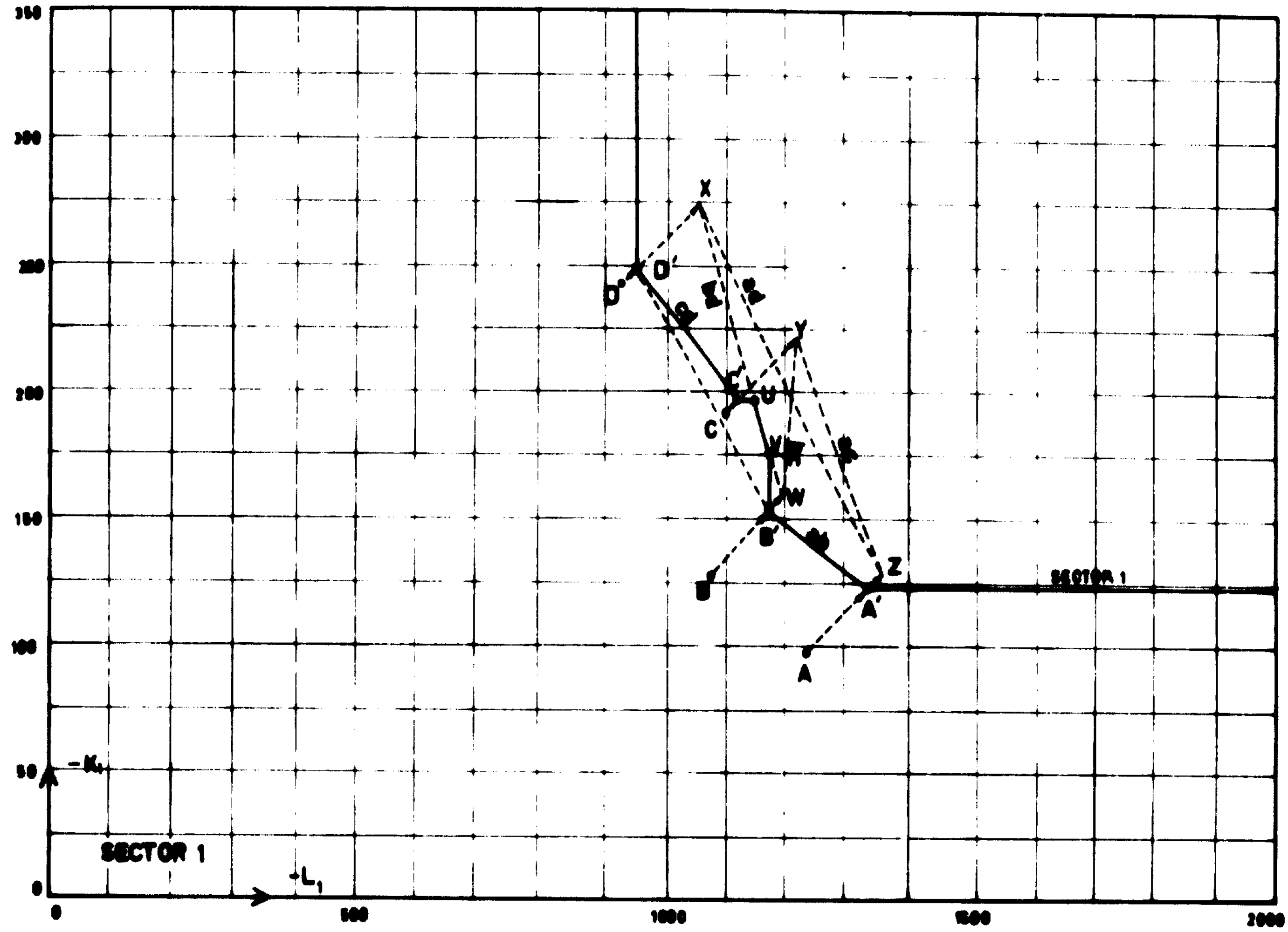

Figure VIII
EXAMPLE OF APEX OCCURRING IN LOCAL INDENTATION

correct optimum in figure VII, as verifiable graphically or by simple algebra, is a capital surplus of 79·9 units that occur at *B'J*. (By comparison, the distance between the sectoral iso-product curves at a line passing through *A'* is only 77·5 units.) The approximation, on the other hand, will yield the overestimate of 86·3 units where the capital surplus is estimated as the vertical distance between *B'* and the *E'F'* apex-to-apex connecting line. The reason for the overestimate is that the approximation ignores the indentation occurring between *E'* and *F'*; i.e., it does not take into account the fact that *three* rather than two fixed costs have to be incurred when complexes *A'* and *F'* are correctly averaged. Note that the indentation will be ignored even when it contains an apex, as at *C'* in figure VIII, because this apex will never be identified. Note also that while in the present case the approximation attains its optimal value at the same combination of complexes as the correct optimum, this cannot be generally expected.

(*b*) Integer programming within the sectoral problems is essential for excluding the possibility of an underestimate. It might be thought that an economy of computation would result if the sectoral integer programming problems were replaced by their linearized versions excluding fixed costs; this would identify new *vertices* from which the corresponding *apices* could be derived by the addition of fixed costs. Such apices, however, would not necessarily lie within the correct iso-product line; they might be dominated by other apices and could lead to an underestimate.

(*c*) The approximation will be good to the extent that non-convexities are weak, i.e., to the extent that local indentations are small in comparison with changes of the objective function corresponding to different basic solutions of the linearized master problem; in other words, to the extent that the apex-to-apex connecting line stays close to the true iso-product line. Closeness is measured in reference to a feasible area which is convex in the large and has only small local non-convexities. Note that the graphical representation permits an intuitive appraisal of the relative roles played by convexity-in-the-large versus non-convexity-in-the-small.

(*d*) Such a situation is likely to arise when fixed costs are small in relation to the changes of variable cost over the averaging ranges, or when the fixed costs of many common activities are shared among neighbouring complexes that differ only slightly in activity composition.

(*e*) Another important situation of this kind arises when fixed costs in a sector are incurred stepwise, i.e., when projects with given fixed costs are limited to a maximum scale, beyond which the fixed cost has to be duplicated. This reduces the size of the abrupt increase in correct averaged costs near the apices and brings the iso-product line within a fraction of the distance from the apex-to-apex connecting line that prevails when fixed costs have to be incurred in a single step (see annex).

(*f*) The computation will be efficient to the extent that the sectoral integer programming problems are small or have a special structure that renders them easy to deal with. Approximate solutions to these sectoral optimizations are acceptable provided they are *dual feasible*, i.e., that they constitute overestimates of the sectoral optima. Algorithms employing cutting planes satisfy this requirement and may thus be terminated after a reasonable number of steps. The approximate sectoral solutions are then available for subsequent iterations of the master problem. As such algorithms show rapid progress initially and often slow down critically near the optimum, the possibility of using the results of runs of limited length might be valuable.

Approximation 2 (unaveraged apices)

In figure IX the unaveraged apices are shown in relation to the correct iso-product line. An approximation to the iso-product line can be pieced together from these apices by adding vertical and horizontal extensions corresponding to free labour and capital disposal activities. In other words, whereas in approximation 1 we formed apex-to-apex connecting lines which gave the appearance that the apices could be averaged in a straight linear fashion, the approximation 2 discards the tool of averaging altogether and simply disposes of labour and capital not required by one apex or another in a given solution. As a result, solutions are restricted to one complex in each sector. The characteristics of this approximation are the following:

(*a*) It always yields an *underestimate* of the potimal value of the objective function, for two reasons: first, because it ignores the possibility of legitimate averaging; and second, because it generally operates with an incomplete list of apices if the problem is large. In the present case the

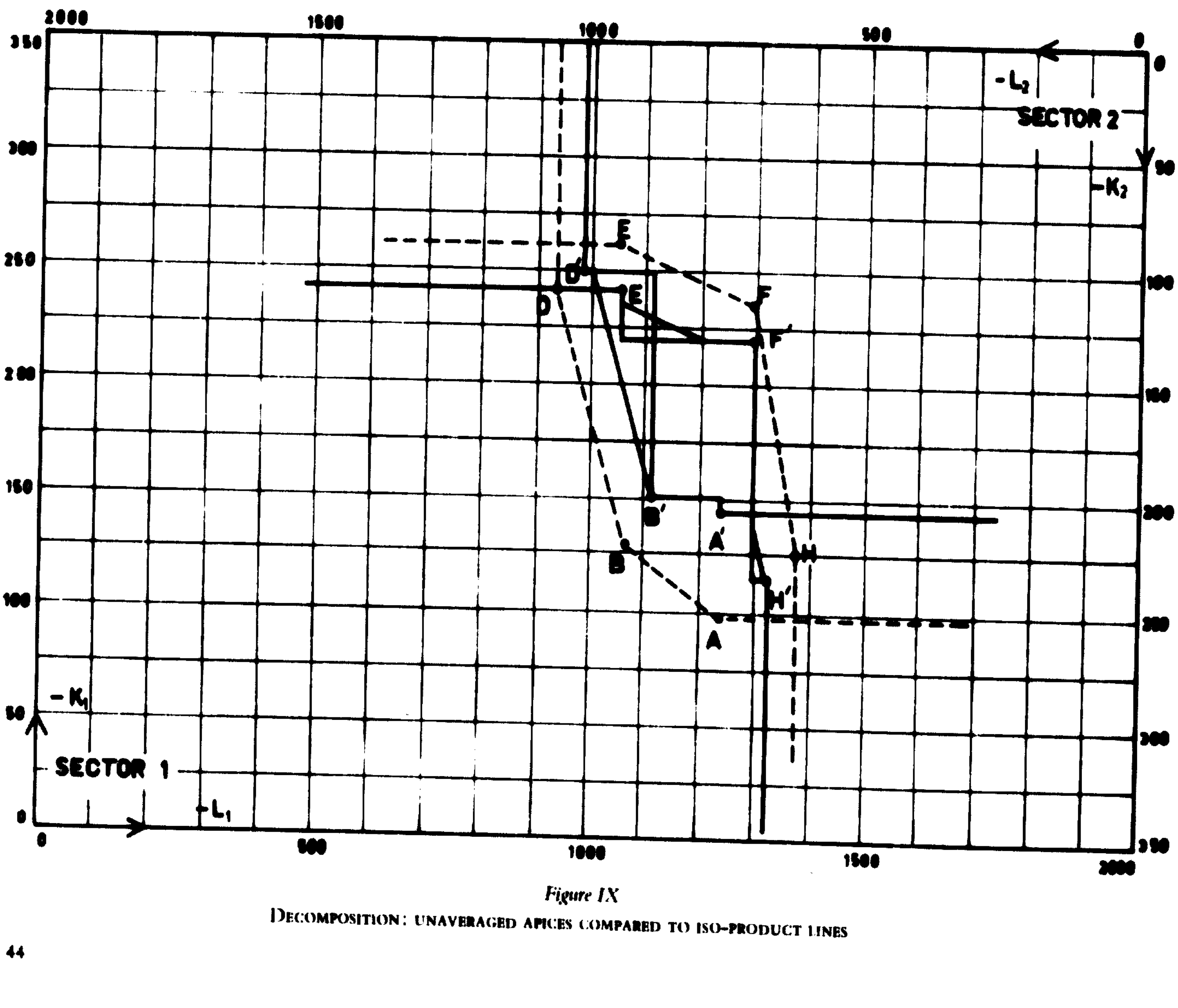

Figure IX

DECOMPOSITION: UNAVERAGED APICES COMPARED TO ISO-PRODUCT LINES optimum occurs using complexes A^1 and F^1, and yields an estimate of 77·5 as against a correct optimum of 79·9 units.

(*b*) The master problem is now an integer programming problem which does not yield useful prices for defining sectoral objective functions.

(*c*) Individual apices may be generated in any convenient way; e.g., by means of simultaneously undertaking the first kind of approximation (apex-to-apex connecting lines), or by linearizing the sectoral problems.

(*d*) Approximation 2 is good whenever non-convexities are large in relation to changes in the objective function corresponding to widely separated solutions; in other words, when the sectors are characterized by a few major indivisibilities. The reasons for this are that in the case of large non-convexities not much is lost by refraining from averaging and the number of apices contributing to the correct iso-product line in any sector is necessarily smaller. Thus the apices are relatively easier to identify on the basis of empirical considerations which are likely to be well known to planners familiar with the sector, and therefore the possibility of missing significant apices is greatly reduced.

(*e*) The computation will be efficient to the extent that the master integer programming problem is of manageable size. If an approximation is required for the master problem, it should be the primal-feasible kind in order to conserve the character of an underestimate.

In sum, the two approximations are complementary. Taken together, they yield both an upper and a lower bound on the value of the optimal solution; in addition, each tends to be close in cases with opposite characteristics. The first approximation tends to be close when the feasible area within a sector is convex in the large and has only small local non-convexities, while the second approximation tends to be close when a sector is characterized by a few major indivisibilities. It is noteworthy that present practical methods of coping with non-convexities in economics tend to run in the direction of these two approximations. Thus in the case of small non-convexities, an attempt is made to define some reasonable average cost and price that will take into account the presence of fixed costs, while in the case of major indivisibilities the operation of the price system is invoked only after quantitative decisions have been taken in regard to these indivisibilities on other than pricing criteria.

The decentralized decision-making process, using these two approximations jointly to simultaneously obtain the upper and lower bounds of the optimal solution, operates in the following fashion. The *starting point* is a feasible basic solution to the linearized master problem; in the present example, this can be provided by a single complex in each sector, together with the labour-disposal activity. The upper decision-making level calculates the prices corresponding to this initial solution and transfers them to the sectors. The *sectors* regard these prices as parameters and optimize their integer programming problems at the given prices; then they pass the combined fixed and variable labour and capital requirements of their optimal solutions to the upper level. The *upper level* checks these factor requirements against the current shadow prices of the linearized master problem including the sectoral "subcontracting fees" (see footnote 26). If no profits occur, approximation 1 has terminated and the current solution of the linearized master problem furnishes an *overestimate* of the correct non-convex optimum, otherwise the profitable complexes are included in the linearized master problem and a new trial solution is computed. As long as profits are present, however, there can be no assurance that the current solution is an overestimate. The upper decision-making level may solve in the course of every iteration an integer programming problem, constructed from the currently available sectoral complexes on the principles of approximation 2. If undertaken, this computation furnishes at every stage an underestimate of the correct non-convex optimum, but it is not necessary to perform the computation until approximation 1 terminates, because the stage-by-stage results are not required for later operations.

An important feature of these stage-by-stage underestimates is that each of the corresponding solutions is feasible, and if the iterative process is broken off at that stage the solution will yield a decision (plan) which can be translated into practice with a known payoff. This is not true of the stage-by-stage solutions of approximation 1; if one of these solutions is translated into practice there is no way of predicting, from the information available to the upper decision-making level, what the actual payoff will be. In other words, if the upper decision-making level instructs the sectors to utilize given sectoral complexes with prescribed weights corresponding to a particular solution to the master problem, the resulting payoff is uncertain. This uncertainty carries over even into the optimal solution obtained by approximation 1. It is known that the latter solution gives an overestimate of the payoff, but when translated into practice there is no assurance that it will yield an actual payoff that is superior to the underestimate provided by approximation 2.

In view of this situation it is useful to introduce the *correct averaging procedure* as an auxiliary feature of the decentralization mechanism. As indicated in more detail in the annex, the correct averaging procedure requires slightly more information than approximations 1 and 2. In addition to the combined factor inputs at the apices, it requires a list of fixed costs incurred at each apex so that *all* fixed costs characterizing both (or several) apices may be included in the correct average. Given this additional information the decentralization mechanism can be strengthened in the following ways:

(*a*) As regards approximation 1, given any feasible solution to the linearized master problem, the actual payoff of this solution in the non-convex system can now be calculated. Moreover, this payoff is a firm underestimate of the optimal payoff in the non-convex system. Thus a solution obtained at any stage of approximation 1 can be translated into practice with a known payoff; in particular, the optimal solution for approximation 1 will now offer both an upper and a lower bound on the optimal payoff of the non-convex system.

(*b*) As regards approximation 2, given a list of apices obtained through the iterations of approximation 1 or otherwise, a master integer programming problem can now be formulated which will allow a solution to be found with a closer approach to the optimal payoff of the non-convex system. Where previously only the straight combination of unaveraged complexes has been permitted, averaging now becomes possible. The solution will still be an *under*estimate since the list of apices is generally incomplete.

THE ROLE OF PRICES IN NON-CONVEX TWO-LEVEL DECISION SYSTEMS

Whichever way the approximate solution to the non-convex problem is identified, the question remains how the upper decision-making level can put such a solution into effect and what role a price system might pay under such circumstances.

Marginal-cost pricing of the connecting resources is consistent with optimal resource allocation. It has already been shown that the slopes of the correct averaging lines (figure V) represent marginal rates of substitution between labour and capital. The same interpretation can also be extended to the horizontal and vertical line segments that are used for eliminating inefficient stretches from the iso-product line (compare figures V and VI). Horizontal line segments represent labour disposal; i.e., over these segments it is more efficient to use a single complex in a sector than to average two complexes, even though the use of a single complex entails the presence of some unused labour. Vertical line segments are similarly obtained by replacing an inefficient averaging of complexes by a single complex: in this case, the capital saving shows up as a net gain that is available only at a single point along the continuum of labour allocations. Thus the marginal rate of substitution in these two cases is zero and infinity, respectively. With this extension the slope of the iso-product line can be interpreted as a marginal rate of substitution at every point where such a slope is defined.

At a local optimum (and thus necessarily at the global optimum as well) the relationship between the marginal rates of substitution of any two connecting resources for any two sectors is a straight extension of the neo-classical efficiency conditions, from the usual smooth iso-product functions to the present angular ones.[32] If the upper decision-making level sets the price ratios of connecting resources to the marginal rates of substitution prevailing at the global optimum,[33] there will be no incentive for further marginal resource reallocations *from the point of view of the system as a whole.*

The indeterminacy of decentralization and control by means of purely price-type instruments that has been observed in linear systems will be present to an even stronger degree when the optimization of sectoral payoffs under marginal-cost pricing is applied to non-convex systems. By sectoral payoff we mean the value of the connecting resources, in the present case, the cost (negative value) of labour and capital. Though, in a linear system a given price ratio will generally sustain an optimum in the sense that at this price ratio no movement away from the optimum will appear advantageous to any of the sectors (even though this optimum will not be attained without the intervention of quantitative controls), in a non-convex system a set of marginal-cost prices will not sustain even the optimum in any stable sense. In a linear system the sectoral payoffs are maximized at the marginal-cost prices corresponding to the over-all system optimum. In a non-convex system, on the other hand, marginal-cost pricing at the system optimum will in some sectors lead only to stationary ranges or points in the payoff,[34] such as the stationary range along the points of the straight-line segments XJY of the iso-product curve of sector 2 passing through J (figure VII). Note that at the labour price set by the slope of this segment the payoff of sector 2 at J is actually at a minimum; alternate minima occur along the entire stretch XY of the straight-line segment. Movement away from J, either to E' or F', would improve the payoff of sector 2. Marginal-cost pricing of the connecting resources can be said to sustain the global optimum $B'J$ only in the limited sense that at such prices sector 2 will be indifferent to *small* local movements along the iso-product curve. However, given the ability of the sector to consider longer-range adjustments to E' or F' (which is certainly a reasonable supposition in the case of two-level planning or decision systems), marginal-cost pricing alone will no longer suffice to sustain even the global optimum of the system.

Thus in non-convex systems where the sectoral values of the connecting resources are to be maximized under marginal cost pricing there is a constant tendency for some sectors to abandon the position required for the system

[32] I. M. D. Little (1950), page 127, summarizes the relevant condition thus: "... the ratio of the marginal products of any two factors of production must be the same for every good in the production of which they both co-operate". Replace the ratio of marginal products by the corresponding marginal rates of substitution and consider a differential reallocation of labour from sector 2 to sector 1. Then the marginal rate of substitution dk/dl in sector 1 can be interpreted as the corresponding decrease in the use (increase in the surplus) of capital and conversely in sector 2. Maximization of the surplus of capital requires $MRS_1 < MRS_2$ when labour allocation to sector 1 is slightly less than that at the optimal solution and $MRS_1 > MRS_2$ when labour allocation to sector 1 is slightly greater than that at the optimal solution. In these formulas MRS is taken as inherently negative. It can readily be verified by reference to figure VII that the global optimum at $B'J$ and the alternate local optima near MN satisfy these conditions.

[33] The marginal rate of substitution will be undefined at the optimum for some of the sectors, e.g., in figure VII at point B' for sector 1. By coincidence it might happen that the same condition holds for a resource in all sectors; in this case more than one price ratio is consistent with the optimum.

[34] Arrow and Hurwicz (1960), pages 88–90.

optimum, and this tendency has to be counteracted by specific quantitative controls such as fixed resource allocations. The practical consequences of the introduction of such quantitative controls are not greatly different from the effects of such controls in linear systems; in this regard non-convexities merely reinforce the control requirements already manifest in linear systems.

In non-convex multi-level decision systems with maximization of sectoral payoffs under marginal-cost pricing for the connecting resources, there will always be external economies and diseconomies linking the sectors. The tendency of some sectors to abandon the position required for system optimum is due to the possibility of improving sectoral payoffs with large readjustments at current marginal-cost prices. But the improvement for the sector would be obtained at the expense of the deterioration of the system as a whole, because the factor reallocations required for the readjustment of the sector in question would leave other sectors with a loss greater than the gain of the first sector. *Marginal-cost pricing furnishes a reliable measure of system-wide repercussions only for differential readjustments near a local optimum,* not for the longer-range readjustments considered here. Thus in the course of such longer-range, non-marginal readjustments of a sector, external economies or diseconomies of the technical variety will come into play: it is as though the sector originated un-priced services or disservices that affect the efficiency of system-wide resource allocation. The question arises as to what the exact nature of these un-priced services or disservices is. Similar questions have puzzled generations of economists concerned with the analysis of externalities.[35]

A key to the problem is furnished by the cutting-plane technique of solving integer programming problems (see annex). When new linear constraints of the proper kind are introduced into linear programming problems whose variables are restricted to integer values, the resulting enlarged linear programming problem will yield an optimal solution identical to the integer solution of the original problem. The enlarged problem is convex, and all constraints, including the newly defined constraints, have proper shadow prices in the optimal solution. *The "resources" corresponding to the newly defined constraint,* unpriced in the original problem, *are the missing services and disservices linking different sectors.*[36] In general there is no unique way of defining new constraints;[37] thus the "missing resources" do not have a definite identity of their own and are just shadowy reflections of the underlying non-convexity of the system. No wonder they have persistently eluded being defined by economists. The search has been all the more frustrating because in some essentially convex systems exhibiting technical externalities, the un-priced resources responsible for the latter are relatively easy to identify (lighthouses *vs.* ships, sparks from railroad engines *vs.* lumber tracts, smoke nuisance, etc.)

Decisions involving the incurrence of fixed costs can generally not be decentralized by a price system and the maximization of sectoral payoffs, except in an approximate fashion. The fixed-cost incurrence activities, even though associated with individual sectors, carry over from the latter into the non-convex master problem involving the correct averaging of a complete list of known feasible basic solutions (compare annex, tables 2 and 3); thus the correct averaging of complexes requires information on sectoral fixed costs, activity by activity. In other words, while the sectors in a linear system are free to choose all aspects of their own technology under a set of centrally announced prices for the connecting resources and are subject to quantitative controls only in regard to a selection from among their alternate optima, in non-convex systems the optimal allocation of resources is contingent on referring all detailed fixed-cost incurrence decisions to the planning centre.

An *approximate* decentralization is, however, possible following the principles of approximation 1. There it was shown (see also annex) that a linearized master problem that handles individual *complexes* as though they were perfectly divisible will yield a set of prices which will guide the system to a satisfactory sub-optimal solution, provided that the deviation of the apex-to-apex connecting lines from the correct iso-product lines is within tolerable error limits. The latter is more likely to be the case when fixed costs can be incurred stepwise in the individual activities rather than requiring all-or-nothing decisions (see annex, figure I). Under such an approximation the individual sectors are again free to choose their own technology *including the assortment of fixed costs to be incurred,* but this gain in decentralization is achieved at the expense of some blurring of the optimal resource allocation for the system as a whole. If, however, the size of the steps by which fixed costs are incurred becomes progressively smaller, the approximation in the limit approaches an exact solution-finding procedure for the case of perfect divisibility, and average factor costs defined by the apex-to-apex connecting line become true marginal costs. The same result also obtains when in the long run fixed costs can be proportionally adjusted to required capacities that are continuously

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[35] For a recent review, see Whinston (1962).

[36] The coefficients of the new constraints are derived from the coefficients of the original constraints in such a way that zero coefficient in the original constraints imply zero coefficients in the new constraints. Thus the new constraints will leave unaffected the sectoral partitioning of a matrix such as that of table 5 or table 6 in the annex. How can we then assert that the new constraints create externalities *linking* different sectors? Even though the new constraints do not possess *coefficients* interlinking the sectors, the *derivation of the new constraints depends on the coefficients of the connecting resources* which of course do link all sectors. In Gomory's fractional method the new constraints depend on the non-integer solution of the (progressively restricted) original problem while in his all-integer algorithm (1960) the maintenance of dual feasibility in the course of his lambda-transformations depends on the coefficients of the objective function and, due to the paramount importance of degeneracy and lexicographic criteria, on the coefficients of the other connecting resources as well.

[37] Gomory and Baumol (1960).

variable: the situation is then an exact analogue of the well-known textbook case of a long-run envelope line derived from the capacity points of linear short-run total-cost curves with fixed costs, each of which has a smaller slope than the envelope (see annex, figure 1*b*).[38] With small but finite fixed cost steps, the apex-to-apex connecting line will yield *exact* estimates of resource requirements at a number of lattice points along the line if at these points all fixed-cost increments already incurred are operating at capacity.[39] At such lattice points, if any exist, the amount of fixed costs actually incurred will be indistinguishable from a perfect long-run adjustment with continuous variability. Such lattice points are most likely to occur in practice when only one of the complexes to be averaged has fixed costs.[40]

If the decentralization of fixed-cost incurrence decisions is altogether abandoned, marginal-cost pricing of the connecting resources will permit the decentralization of other sectoral decisions. If the upper decision-making level provides a list of fixed costs to be incurred in the sectors, the activities whose fixed costs have not been incurred will be inactivated. The other activities will now jointly define a *linear* two-level decomposition problem whose solution implies marginal-cost pricing of the connecting resources. Such a decision strategy is often suitable, e.g., for plant location problems where after the selection of active plants the remaining production-and-transport problem is convex and leads to a well-defined system of shadow prices. The central selection of active plants can be undertaken by suitable approximations such as those described in the section on the decomposition principle in non-convex systems.

In this case the decision process is divided into two stages. During the first stage information is interchanged between the upper and lower decision-making levels; as a result a programme emerges that represents a target decision (plan) for all levels of the system. In stage two this plan is to be implemented. Following the announcement of active plants by the centre, and reduction of capital and labour availabilities by the fixed amounts already committed, further implementation can follow the trial-and-error adjustments of a linear decomposition system, held within reasonable bounds by the quantitative controls that are always needed in linear systems for the weighting of alternate sectoral optima in some of the sectors at the given prices.

"*Average-cost" pricing of the connecting resources is preferable to marginal-cost pricing, to the extent that the decentralization of fixed-cost incurrence decisions is of practical concern.* In this context "average-cost" pricing of the connecting resources is taken to mean prices that correspond to the slopes of the apex-to-apex connecting lines.

In many practical problems of multi-level decision-making, e.g., in national economic planning, the decentralization of fixed-cost incurrence decisions is of the utmost importance. In national planning large numbers of investments have to be identified by class of economic activity, time period and location, and it is desirable that the central planning level be relieved of all but the most essential of these decisions. Average-cost pricing of the connecting resources (as defined above) by the upper decision-making level will permit trial-and-error adjustments in the course of plan implementation that follow the principles of approximation 1. These adjustments, like the ones of a linear system, have to be held within reasonable bounds by quantitative controls which *prescribe* the weighting of given sectoral complexes, even though, in contrast to *the linear case*, this may lead to a reduction of some sectoral payoffs.

In this, as in the previous case, the decision process will generally have to be divided in two stages. The first stage is required in order that the degree of error of resource allocation, inherent in the decentralization of fixed-cost incurrence decisions, may be judged. If this error is tolerable, the second stage can follow with its trial-and-error adjustments in the course of plan implementation.

More generally the first stage permits a judgement concerning which fixed costs are to be centrally prescribed and which are to be left for decentralized decision making. It is entirely possible to prescribe or to suppress one group of fixed costs while another group is left open for decentralized decisions. In this case the prices of the connecting resources will be "average-cost" prices as before, but with certain fixed costs (those that have been prescribed) omitted from the complexes, and with certain complexes (those that contain activities with suppressed fixed costs) omitted altogether from consideration. As in approximation 1, the sectors will maximize their payoffs, taking into consideration only those fixed costs whose incurrence has been left open. Resource availabilities are again reduced by the fixed costs already committed. At every stage of plan implementation the trial-and-error adjustment is kept within bounds by quantitative controls.

In a two-level decision system, where sectoral decisions occur as a unit, the question of pricing for individual activities can be left open. Given the non-convexity of fixed-cost incurrence operating *within* as well as *between* sectors, there will be externalities (and corresponding unpriced "missing resources") at the sectoral level as well as at the level of the system as a whole; therefore, no simple

[38] These cases are compatible with perfect competition that would prevail in the *undecomposed* system as a whole, without any need for upper or lower decision-making levels. Here all decisions are decentralized to the individual activity within a sector; this activity is a composite of the production and fixed-cost-incurrence activities, with all resource coefficients corresponding to operation at capacity level (see model of table 5, annex). This is a limiting case of competitive supply with U-shaped average cost curves; with linear total costs up to capacity, the left leg of the "U" is a hyperbola, the right leg is vertical, and the level of minimum average costs is determined (besides variable costs) by average fixed costs at capacity operation.

[39] Fixed costs, common to both complexes, that are incurred in a single step at the scales of these complexes, need not meet this requirement.

[40] Except for fixed costs of the kind specified in the previous footnote.

pricing prescriptions can be expected. In considering intra-sectoral pricing rules, the possibility of further decentralization must be kept present; moreover, on practical grounds the setting of prices to cover *average* costs approximately at planned output levels is to be strongly favoured.

Conclusion

The investigation of the properties of two-level decision systems with angular decomposable structure made non-convex by the inclusion of fixed costs yields some preliminary insights into the structure of more general non-convex economic systems. Thus it is apparent that non-convexity usually will give rise to externalities, but the converse cannot be asserted. Moreover, an economic system such as that of a present-day predominantly private-enterprise industrial economy can operate with a reasonable degree of efficiency in spite of its pervasive elements of non-convexity, provided that (*a*) highly indivisible decisions are subject to some kind of rational centralized deliberation independent of the market, and (*b*) smaller irregularities are adjusted by a price system that is based on average costs near the highest efficient scale of operation. It is equally clear that these preconditions are only imperfectly satisfied in practice and that a properly functioning decentralized planning system can contribute substantially to coping with the problems of efficient resource allocation.

ANNEX

Notation[a]

The schema given in table 1 (text), following Tucker (1963), is a succinct joint representation of two systems of linear equations. One system arises by forming the inner products of the vector

$$X = [X_0, \dots, X_N]$$

with the rows of the matrix A, the matrix of numerical coefficients in table 1 (text), and setting each inner product equal to the corresponding components of

$$\sigma = \begin{vmatrix} \sigma_0 \\ \cdot \\ \cdot \\ \cdot \\ \sigma_M \end{vmatrix}$$

This leads to a system of $M + 1$ linear equations in the N variables X_j:

$$\begin{array}{ccccc} \sigma_0 & = a_{00}X_0 & + \dots + & a_{0N}X_N \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sigma_M & = a_{M0}X_0 & + \dots + & a_{MN}X_N. \end{array}$$

The second system arises by forming the inner products of

$$Y = \begin{vmatrix} Y_0 \\ \cdot \\ \cdot \\ \cdot \\ Y_M \end{vmatrix}$$

with the columns of A, and setting each equal to the corresponding component of

$$-L = [-L_0, \dots, -L_N].$$

This yields a system of $N + 1$ linear equations in the variables Y_i:

$$\begin{array}{ccccc} -L_0 & = a_{00}Y_0 & + \dots + & a_{M0}Y_M \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -L_N & = a_{0N}Y_0 & + \dots + & a_{MN}Y_M. \end{array}$$

In table 1 (text), the negative variables $-L_j$ have been replaced by positive variables π_j. Tucker (1963, pages 8–9) shows how the systems of linear equations given in the schema in homogeneous form can be employed in non-homogeneous form by setting X_0 and Y_0 equal to unity to treat dual linear programmes. Here σ_0 becomes the maximand of the primal problem and $-L_0$ becomes the minimand of the dual problem, with all other variables required to be non-negative. When we replace the $-L_j$ variables by π_j variables, the minimand becomes π_0 and the other π_j variables are required to be non-positive. In the economic interpretation of the problem, the L_j are losses while the $-L_j + \pi_j$ are profits; the X_j are activity scales, the Y_i shadow prices of resources, and the σ_i unused resource surpluses.

A full algebraic expansion of the problem summarized in table 1 (text) is given in the table on page 50.

The linear decomposition problem

The *master problem* (extremal sub-problem) and one of the two *sectoral sub-problems*, following the decomposition method of Dantzig and Wolfe (1961), are given in table 1 (below), again using Tucker's condensed schema. The interpretation agrees with the discussion above, except for a slight modification in the case of the sectoral sub-problem. Here the maximand z_1 is a weighted average of the two top rows with p_K and p_L employed as constant weights; in particular p_K is always set to unity. The resource surpluses characterizing the two top rows will be negative (as capital and labour are not produced and as there are no exogenous sectoral supplies) and are not subject to the non-negativity constraints applicable to the other resource surpluses. The constant weights p_K and p_L are set in the course of each iteration to the last solution values of the same variables in the master problem. The sub-problem for the second sector (not shown) can be derived from the original problem by analogy.

[a] Quoted, except for a change in notation, from Tucker (1963), pages 1–2.

Primal problem:										
Max!	σ_0	350	$1{\cdot}1X_1$	$1{\cdot}25X_2$	$3{\cdot}0X_3$	$7{\cdot}5X_4$	X_5	$2{\cdot}5X_6$	$0{\cdot}6X_7$	$3{\cdot}0X_8$
Subject to	σ_1	2000	$12{\cdot}5X_1$	$7{\cdot}5\ X_2$	$6{\cdot}0X_3$	$7{\cdot}0X_4$	$15{\cdot}0X_5$	$5{\cdot}0X_6$	$4{\cdot}0X_7$	$11{\cdot}0X_8$
	σ_2	50	X_1	X_2	$0{\cdot}5X_3$	$0{\cdot}2X_4$				
	σ_3	50		$0{\cdot}25X_2$	X_3	X_4				
	σ_4	25					X_5	X_6	$0{\cdot}8X_7$	
	σ_5	25					$0{\cdot}2X_5$	$0{\cdot}5X_6$	X_7	X_8
and	$\sigma_i \geqslant 0, i\ \ 1, \ldots, 5$									
	$X_j \geqslant 0, j\ \ 1, \ldots, 8$									

Dual problem:							
Min!	π_0	350	$2000\ Y_1$	$50\ Y_2$	$50\ Y_3$	$25\ Y_4$	$25\ Y_5$
Subject to	π_1	$1{\cdot}1$	$12{\cdot}5Y_1$	Y_2			
	π_2	$1{\cdot}25$	$7{\cdot}5Y_1$	Y_2	$0{\cdot}25Y_3$		
	π_3	$0{\cdot}3$	$6{\cdot}0Y_1$	$0{\cdot}5Y_2$	Y_3		
	π_4	$2{\cdot}5$	$7{\cdot}0Y_1$	$0{\cdot}2Y_2$	Y_3		
	π_5	$1{\cdot}0$	$15{\cdot}0Y_1$			Y_4	$0{\cdot}2Y_5$
	π_6	$2{\cdot}5$	$5{\cdot}0Y_1$			Y_4	$0{\cdot}5Y_5$
	π_7	$0{\cdot}6$	$4{\cdot}0Y_1$			$0{\cdot}8Y_4$	Y_5
	π_8	$3{\cdot}0$	$11{\cdot}0Y_1$				Y_5
and	$Y_i \geqslant 0, i\ \ 1, \ldots, 5$						
	$\pi_j \leqslant 0, j\ \ 1, \ldots, 8$						

Depending on p_L/p_K, different optima to the sectoral sub-problems will be attained.[b] The possible optima include, for the present problem, vertices *A*, *B*, and *D* for sector 1, and vertices *E*, *F*, and *H* for sector 2. Designate the total capital and labour requirements of any of these optima by K_w, L_w, where *w* is the index of a vertex. The requirements appear in the master problem (table 1(*a*)).

In the master problem $\lambda_A, \ldots, \lambda_H$ are variable weights to be attached to each of the sectoral vertex solutions. These weights have to add up to unity for convex combinations of vertices, as expressed by the constraints of the third and fourth row. Note that the new resource surplus variables σ_9 and σ_{10} corresponding to these constraints are *artificial*; i.e., they are introduced only formally, since they are required to be exactly zero in the optimal solution. The other two resource surpluses are the same as encountered in the original problem and refer to capital and labour, respectively. The capital surplus is maximized. The variables of the dual problem are the shadow prices p_K, p_L, p_1, p_2 associated with capital, labour, and the two convexity constraints. The shadow price for capital is identical with the variable Y_0 in the original problem and is set equal to unity; the shadow price for labour is identical to Y_1. The shadow prices p_1 and p_2 can be interpreted as "subcontracting fees" as discussed in footnote 26 of the text. The variables $\pi_A, \ldots, \pi_H$ are profits at shadow prices, associated with the use of each vertex (complex of activities). The dual minimand π_0 can be interpreted as the net valuation of exogenously given supplies and demands at shadow prices, where the (1) entries in the exogenous column stand for the net exogenous demands of sectoral resources.

[b] The optima may be extreme-point (vertex) or homogeneous solutions (Dantzig and Wolfe, 1961). Homogeneous solutions indicate that the maximand of the sub-problem may be expanded without limit; in other words, the specific sectoral resource constraints do not preclude such an expansion. If such a situation occurred in the full problem, it would indicate that the problem was unbounded; but the solutions to the sectoral sub-problems are also subject to the constraints on the connecting resources, and thus homogeneous sectoral solutions are permissible (Dantzig and Wolfe, 1961, pages 773–774). None such occurs in the present problem.

In the above formulation, all vertex solutions (efficient complexes) are included in the master problem. If, in fact, all of these were present from the very beginning, the solution to the master problem would at once yield the over-all optimum. The algorithm operates, however, with only a partial list of such vertices which initially define only a single feasible starting solution. At any stage of the algorithm the current optimum to the master problem yield a set of shadow prices. At these prices, all vertices with positive λ_w weights have zero profits, while other vertices have negative profits; no positive profits can occur at any such an optimum.

In order to test whether the current optimum to the master problem is also an over-all optimum, an attempt is made to find a new vertex that will show a positive profit at current shadow prices. Since p_1 and p_2 are given, a profitable new vertex *w* must have the highest possible algebraic value for the expression

$$(\ p_K . K_w \ \ p_L . L_w),$$

where p_K and p_L are also given. The sectoral sub-problems select the vertex which maximizes the above expression in each sector. If the algebraic sum of p_1 or p_2 and this maximum is *positive* for a sector, vertex *w* is profitable and the current optimum to the master problem is not an over-all optimum. The new vertex is then included in the list of known vertices, and the optimization for the master problem is repeated. In the contrary case the over-all optimum has been attained.[c]

THE NON-CONVEX DECOMPOSITION PROBLEM

In the presence of fixed costs the original problem has to be expanded as shown in table 2. The fixed costs of each activity X_j shown in table 1 (text) are introduced as the capital and labour inputs associated with new activities X_j^*. The levels of these new activities are tied to the level of each corresponding original activity X_j by means of proportionality constraints that force the

[c] For details including homogeneous sectoral solutions and an algebraic exposition applicable to *n* sectors, see Dantzig and Wolfe (1961).

Table 1

DANTZIG-WOLFE DECOMPOSITION METHOD FOR LINEAR SYSTEM

(a) MASTER PROBLEM

		Sector 1			Sector 2				
		0	0	0	0	0	0		
	Min!	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$		
	π_0	π_A	π_B	π_D	π_E	π_F	π_H		
Max! $\sigma_0 =$	350	$-K_A$	$-K_B$	$-K_D$	$-K_E$	$-K_F$	$-K_H$	* p_K y_0 (= 1)	Capital
$0 \leq \sigma_1 =$	2000	$-L_A$	$-L_B$	$-L_D$	$-L_E$	$-L_F$	$-L_H$	* p_L y_1	Labour
$0 = \sigma_9 =$	−1	1	1	1				* p_1	
$0 = \sigma_{10} =$	−1				1	1	1	* p_2	
	*	*	*	*	*	*	*		
	X_0 (= 1)	λ_A	λ_B	λ_D	λ_E	λ_F	λ_H		
	Exogenous	Averaging weights for complexes							

(b) SECTORAL SUB-PROBLEM FOR SECTOR 1

			0	0	0	0		
	Min!		$\geq$	$\geq$	$\geq$	$\geq$		
		π_0^1	π_1	π_2	π_3	π_4		
Max! $z_1 =$	$\sigma_0^1 =$	0	−1.1	−1.25	−.3	−2.5	* p_K } Constants	Capital
$-p_K\sigma_0^1 + p_L\sigma_1^1$	$\sigma_1^1 =$	0	−12.5	−7.5	−6.0	−7.0	* p_L }	Labour
$0 \leq$	$\sigma_2 =$	−50	1	1	−.5	−.2	* y_2	
$0 \leq$	$\sigma_3 =$	−50	0	−.25	1	1	* y_3	
		*	*	*	*	*		
		X_0^1 (= 1)	X_1	X_2	X_3	X_4		
		Exogenous	Production activities					

level of each X_j^* to equal or exceed a constant fraction α of activity X_j. The new activities may be interpreted as the fraction of fixed costs actually incurred. Of course the only levels of X_j^* that make economic sense are 0 and 1; thus we impose the constraint that X_j^* has to be an integer, thereby converting the problem into one of integer programming.[d] The constant fraction α is chosen small enough so that it will not drive the value of any X_j^* above unity. With these provisions we have the following chain of interactions between the variables: X_j cannot exceed 0 unless X_j^* rises to at least αX_j; once X_j^* rises above zero, however, the integrality condition takes over and drives it all the way to unity. Thus X_j cannot exceed 0 unless the corresponding fixed cost is incurred in its entirety.

The new variables introduced in table 2 include, besides the X_j^*, also profits π_j^* on these activities. New "resources" also appear corresponding to the proportionality constraints linking the linear production activities with the new fixed-cost activities: the non-negative "resource surpluses" in these rows are designated as σ_j^* and the shadow prices as Y_j^*, where the subscript corresponds to the production activity whose fixed cost gives rise to the proportionality constraint in question.

It should be noted that the primal-dual representation of the integer programming problem in table 2 is incomplete, as the possibility of simultaneously satisfying both primal and dual constraints—while assuring integer values for the fixed-cost

[d] While values exceeding 1 will be excluded by the optimization process itself, the convergence of some integer programming algorithms is improved by an explicit upper bound.

Table 2

FORMULATION OF DECOMPOSITION MODEL WITH FIXED COSTS

		Exogenous	Sector 1								Sector 2								
		Min'	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
		π_0	π_1	π_2	π_3	π_4	π_1^*	π_2^*	π_3^*	π_4^*	π_5	π_6	π_7	π_8	π_5^*	π_6^*	π_7^*	π_8^*	
Max'	σ_0	350	−1.1	−1.25	−.3	−2.5	−3.0	−5.	−15.	0	−1.	−2.5	−.6	−3.	−10.	0	−10.	−5.	* y_0 (−1) Capital
0	σ_1	2000	−12.5	−7.5	−.6	−7.	0	−50.	0	0	−15.	−5.	−4.	−11.	0	−50.	0	0	* y_1 Labour
0	σ_2	−50	1	1	−5.	−.2													* y_2 Sector 1 special resources
0	σ_3	−50	0	−.25	1	1													* y_3 Sector 1 special resources
0	σ_1^*	0	−a				1												* y_1^* Sector 1 proportionality construction
0	σ_2^*	0		−a				1											* y_2^* Sector 1 proportionality construction
0	σ_3^*	0			−a				1										* y_3^* Sector 1 proportionality construction
0	σ_4^*	0				−a				1									* y_4^* Sector 1 proportionality construction
0	σ_4	−25									1	1	−.8	0					* y_4 Sector 2 special resources
0	σ_5	−25									−.2	−.5	1	1					* y_5 Sector 2 special resources
0	σ_5^*	0									−a				1				* y_5^* Sector 2 proportionality construction
0	σ_6^*	0										−a				1			* y_6^* Sector 2 proportionality construction
0	σ_7^*	0											−a				1		* y_7^* Sector 2 proportionality construction
0	σ_8^*	0												−a				1	* y_8^* Sector 2 proportionality construction
		* X_0 (−1)	* X_1	* X_2	* X_3	* X_4	* X_1^*	* X_2^*	* X_3^*	* X_4^*	* X_5	* X_6	* X_7	* X_8	* X_5^*	* X_6^*	* X_7^*	* X_8^*	
		Exogenous	Production activities				0,1 Fixed cost activities				Production activities				0,1 Fixed cost activities				

variables—depends on the introduction of *cutting planes*.[c] The cutting planes appear as additional constraint rows that are implied by the constraints of the problem when integrality requirements are imposed. In the absence of such extra constraints no primal-feasible *integer* solution generally exists that would also satisfy the dual constraints. *All Tucker schemata presented for integer programming problems in this paper must be interpreted with this reservation in mind.*

How does the presence of fixed costs affect the master problem of the purely linear case? Given a list of the capital and labour input requirements (excluding fixed costs) of all feasible basic solutions to the sectoral sub-problems, we also need the fixed costs of all sectoral activites and a specification of the individual activities that are operative in each solution on the list. On the basis of this information we can formulate a master problem for the integer programming case as shown in table 3. All fixed-cost activities X_j^* are explicitly included in the revised master problem, and proportionality constraints are added connecting the level of a complex with the scale of each fixed cost activity required for running that complex. When complexes are averaged, accordingly, these constraints will force the scales of all the X_j^* required for *any* of the complexes above zero, and the integrality constraints for these variables will force them further up to unity. Note that a knowledge of the *combined* variable and fixed input requirements for each connecting resource (capital and labour) associated with a complex is not enough for deriving the over-all optimum; it is also necessary to have correct averaging rules. In the present case the fixed-cost incurrence rules have the effect of such correct averaging rules.

In table 3 as in the linear master problem, each complex is represented by its aggregate *variable* capital and labour requirements that appear in the first two rows. The level of operation of each complex is a variable λ_w that can be interpreted, as in the linear case, as a weight. Weighted averages of complexes are formed by requiring that these non-negative weights add up to unity: this is expressed by the last two rows; these agree with the linear case. Note that in the linear case only the vertex solutions were included, i.e., only efficient feasible basic solutions participated in the master programme. In making the transition from the linear to the non-convex case, due to reasons cited in the text, the possibility cannot be excluded that previously inefficient feasible basic solutions may contribute a segment to the new production possibility frontier. Therefore the complete list of feasible basic solutions has to appear in the master problem. Fixed-cost activities are added with their usual activity levels X_j^* and profit levels π_j^*. The proportionality constraints again give rise to new "resources" whose surpluses and shadow prices are designated by $\sigma_{w,j}$ and $p_{w,j}$, respectively, where w is the index of the complex and j is the index of the respective fixed-cost activity.

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[c] Gomory developed the first successful method of defining cutting planes. A simple derivation of cutting planes for the former algorithm as well as for an all-integer algorithm will be found in Gomory (1960 b). Specific methods for dealing with mixed integer problems were developed by Beale (1958), Gomory (1960 a), and others. For a general survey of available solution and approximation techniques for integer programming see Beale (1965) and Balinski (1965).

Table 3

INTEGER PROGRAMMING MASTER PROBLEM

		π_0	π_A	π_B	π_C	π_D	π_E	π_F	π_G	π_H	π_1^*	π_2^*	π_3^*	π_4^*	π_5^*	π_6^*	π_7^*	π_8^*	
	Min!		$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	
Max!	σ_0	350	−97.5	−128.6	−191	−243.4	−89.3	−115	−1375	−225	−30	−5	−15	−0	−10	0	−10	−5	$* p_K$
$0 \leqslant$	σ_1	2000	−1237.5	−1071.4	−1100	−934.2	−946.4	−705	−787.5	−625	0	−50	0	0	0	−50	0	0	$* p_L$
$0 \leqslant$	$\sigma_{A,1}$		$-a$								1								$* p_{A,1}$
$0 \leqslant$	$\sigma_{A,3}$		$-a$										1						$* p_{A,3}$
$0 \leqslant$	$\sigma_{B,2}$			$-a$								1							$* p_{B,2}$
$0 \leqslant$	$\sigma_{B,3}$			$-a$									1						$* p_{B,3}$
$0 \leqslant$	$\sigma_{C,1}$				$-a$						1								$* p_{C,1}$
$0 \leqslant$	$\sigma_{C,4}$				$-a$									1					$* p_{C,4}$
$0 \leqslant$	$\sigma_{D,2}$					$-a$						1							$* p_{D,2}$
$0 \leqslant$	$\sigma_{D,4}$					$-a$								1					$* p_{D,4}$
$0 \leqslant$	$\sigma_{E,5}$						$-a$								1				$* p_{E,5}$
$0 \leqslant$	$\sigma_{E,7}$						$-a$										1		$* p_{E,7}$
$0 \leqslant$	$\sigma_{F,5}$							$-a$							1				$* p_{F,5}$
$0 \leqslant$	$\sigma_{F,8}$							$-a$										1	$* p_{F,8}$
$0 \leqslant$	$\sigma_{G,7}$								$-a$								1		$* p_{G,7}$
$0 \leqslant$	$\sigma_{G,8}$								$-a$									1	$* p_{G,8}$
$0 \leqslant$	$\sigma_{H,6}$									$-a$						1			$* p_{H,6}$
$0 \leqslant$	$\sigma_{H,7}$									$-a$							1		$* p_{H,7}$
$0 =$	σ_9	−1	1	1	1	1													$* p_1$
$0 =$	σ_{10}	−1					1	1	1	1									$* p_2$
		$* X_0$	$* \lambda_A$	$* \lambda_B$	$* \lambda_C$	$* \lambda_D$	$* \lambda_E$	$* \lambda_F$	$* \lambda_G$	$* \lambda_H$	$* X_1^*$	$* X_2^*$	$* X_3^*$	$* X_4^*$	$* X_5^*$	$* X_6^*$	$* X_7^*$	$* X_8^*$	
		(=1) Exogenous	Averaging weights for individual complexes								0, 1 Fixed cost activities								

The integer programming version of the master problem, when operating on a *partial* known list of complexes, will yield an *underestimate* of the correct optimum.

APPROXIMATIONS

The two approximations described in the text have a reduced need for information. They operate on the *combined* fixed plus variable factor inputs of a complex without recourse to further information on the fixed costs of activities that make up the complex.

Table 4 lists these combined factor inputs for the apices $A', \ldots, H'$, while tables 5 and 6 formulate the approximation methods. Table 5a shows the *linearized* master problem of approximation 1 while table 5(*b*) shows one of the corresponding sectoral *integer programming* sub-problems. Table 6 formulates approximation 2.

In the master problem of approximation 1 (table 5(*a*)) the capital and labour inputs of the individual complexes correspond to the sum of fixed and variable requirements, as can be verified by reference to table 4. The averaging weights are subject to the usual convexity postulate (rows 3 and 4); the fixed-cost activities used for correct averaging (see table 3) are omitted. Note, however, that the factor inputs of a given complex in table 3 do not contain the fixed factor requirements, whereas in table 5(*a*) they do: i.e., complex *B* in table 3 has capital and labour inputs of −128·6 and −1,071·4, respectively, while complex *B* in table 5(*a*) has corresponding inputs of −148·6 and −1,121·4. The effect of this difference is that the model of table 5(*a*) permits the linear combination of apices, i.e., it assumes perfect divisibility of the fixed costs included in each complex; contrariwise, the model of table 3, while giving the same combined factor inputs as table 5(*a*) for each apex (due to the operation of the proper fixed-cost activities whenever any *one* apex is used), in addition enforces the full incurrence of additional fixed costs when complexes are used in combination.

Similarly to the linear master problem of table 1(*a*) the *linearized* master problem of approximation 1 starts out with just one feasible basic solution and generates additional apices as required. The prices for labour and capital, p_K and p_L, calculated from the model of table 5(*a*) in each iteration, are inserted, as in the linear case, into the sectoral sub-problems (compare tables 1(*b*) and 5(*b*)). In table 5(*b*), however, this sub-problem is an integer programming problem. The solution to each sub-problem will be an apex

Table 4

VARIABLE, FIXED, AND COMBINED FACTOR INPUTS FOR COMPLEXES

Sector 1			Sector 2		
Complex	*Capital*	*Labour*	*Complex*	*Capital*	*Labour*
A'	97·5	1,237·5	E'	89·3	946·4
	45	0		20	0
	142·5	1237·5		109·3	946·4
B'	128·6	1,071·4	H'	225·0	625·0
	20	50		10	50
	148·6	1,121·4		235·0	675·0
C'	191·0	1,100·0	F'	115·0	705·0
	30	0		15	0
	221·0	1,100·0		130·0	705·0
D'	243·4	934·2	G'	137·5	787·5
	5	50		5	50
	248·4	984·2		142·5	837·5

contained in the convex hull of the non-convex production possibility function for each sector. When these apices are included in the master problem their combined factor cost may or may not be less than the current *sub-contracting fee* p_i for the sector (see the discussion of the linear case), i.e., either leaving or not leaving a positive profit. As long as there is a profit, the solution to the master problem must be changed by entering any profitable activity into the base; when profits no longer appear, the over-all optimum *to the approximation* has been attained. Since some fixed costs that should have been included in combined complexes may have been suppressed by the linear averaging procedure of the master problem, the optimal solution to this approximation, if not an exact optimum to the integer programming problem, is an overestimate.

Table 6 represents approximation 2. The rows of this table are identical with the top two rows of table 5(*a*); however, while in table 5(*a*) the λ were continuous variables, in table 6 they become 0, 1 variables.

STEPWISE INCURRENCE OF FIXED COSTS

In figure I a section of figure VII (text) corresponding to complexes E and F has been redrawn on a slightly enlarged scale, on the assumption that the fixed costs of the individual activities making up the two complexes can be incurred in several steps. The data are identical to those in table 1 (text) except for the following changes. Activity X_7 incurs fixed costs in units of expenditure of (2·0) for capital and labour, respectively; these fixed costs yield a capacity of up to 8 units of X_7. Activity X_8 similarly incurs fixed costs in units of (1·0) with a corresponding capacity of up to 6 units of X_8. Activity X_5 which occurs in both complexes is assumed to incur its fixed costs all at once, in order to simplify the graph.

As usual point E represents the capital and labour inputs of a complex excluding fixed costs; the costs have to be added on separately to obtain the corresponding apex E'. The addition of $\bar{x}_5$ is represented by a vertical arrow; $\bar{x}_7$ is not added on all at once but in five steps. The scale of X_7 in complex E is 35·714 (see table 1 in text) while in complex F it is 0; thus when complexes E and F are correctly averaged with the weight of complex E varying from 0 to 1, the scale of activity X_7 increases from 0 to 35·714. Since each unit of fixed costs yields a capacity of 8 units, as the scale of X_7 increases along the F–E connecting line the first step of $\bar{x}_7$ is incurred at W (arrow of 2 units pointing vertically downward), and yields a maximum capacity corresponding to point e_1; thereafter another fixed-cost step has to be incurred that will yield capacity up to e_2, etc.; the final step of fixed costs is incurred after e_4 and yields a maximum capacity corresponding to point e_5. Up to this point the cumulative capacity has reached 40 units which is beyond the 35·714 units required for apex E'; at E', however, the cumulative fixed cost incurred is already (10·0) units. Similarly the fixed cost of activity X_8 is incurred in steps; the first of these is shown as an arrow of unit length pointing vertically downward from V: this yields a maximum capacity corresponding to point f_1; thereafter another step is incurred, with maximum capacity at f_2, etc. When the fixed-cost expenditures for X_8 are added to the step function resulting from the stage-by-stage incurrence of the fixed costs of X_7, the shape of the resulting total expenditure curve is quite jagged. Some parts of this curve are, however, dominated by other parts and have to be replaced by horizontal lines representing labour disposal; the black triangles in the plot represent the parts of the curve that are cut away. The final iso-product line runs along the tops of these triangles.

While drawing such an iso-product line to scale requires some care, the fundamental concept is simple and corresponds to the textbook case of linear total cost curves with capacity limits that are proportional to fixed costs (shown for reference in figure I(*b*)). In the latter case the capacity limits occur at points y_i which fall along a straight line OY; in figure I(*a*) likewise capacity limits for X_7 fall along a straight line connecting points W, e_1, e_2, . . . , e_5 and for X_8 along a straight line V, f_1, f_2, f_3, f_4, F'. The apex-to-apex connecting line $E'F'$ in the present case is considerably closer to the sectoral iso-product line than when fixed costs have to be incurred in a single step. For reference the iso-product line for the latter case is added: see line $E'XYF'$. With perfect divisibility the iso-product line becomes the sum of two straight lines, VF' and

Table 5

Approximation I non-convex programming problem

(a) MASTER PROBLEM

	Min!	Sector 1				Sector 2				
		$0 \geq \pi_A$	$0 \geq \pi_B$	$0 \geq \pi_C$	$0 \geq \pi_D$	$0 \geq \pi_E$	$0 \geq \pi_F$	$0 \geq \pi_G$	$0 \geq \pi_H$	
	π_0									
Capital Max! σ_0 =	350	−142.5	−148.6	−221	−248.4	−109.3	−130	−142.5	−235	* p_K y_0 (= 1)
Labour $0 \leq \sigma_1$ =	2000	−1237.5	1121.4	−1100	−984.2	−946.4	−705	−837.5	−675	* p_L y_1
$0 = \sigma_9$ =	−1	1	1	1	1					* p_1
$0 = \sigma_{10}$ =	−1					1	1	1	1	* p_2
	*	*	*	*	*	*	*	*	*	
	X_0 (= 1) Exogenous	λ_A	λ_B	λ_C	λ_D	λ_E	λ_F	λ_G	λ_H	
		Averaging weights for complexes								

(b) SECTORAL SUB-PROBLEM FOR SECTOR 1

	Min!									
	π_0^1	$0 \geq \pi_1$	$0 \geq \pi_2$	$0 \geq \pi_3$	$0 \geq \pi_4$	$0 \geq \pi_5$	$0 \geq \pi_2^*$	$0 \geq \pi_2^*$	$0 \geq \pi_4^*$	
Max! $z^1 = \sigma_0^1 =$	0	−1.1	−1.25	−.3	−2.5	−30	−5	−15	0	* p_K Constants
$-p_K\sigma_0^1 + p_L\sigma_1^1$ σ_1^1 =	0	−12.5	−7.5	−6.0	−7.0	0	−50	0	0	* p_L Constants
$0 \leq \sigma_2$ =	−50	1	1	−.5	−.2					* y_2
$0 \leq \sigma_3$ =	−50	0	−.25	1	1					* y_3
$0 \leq \sigma_1^*$ =	0	$-a$				1				* y_1^*
$0 \leq \sigma_2^*$ =	0		$-a$				1			* y_2^*
$0 \leq \sigma_3^*$ =	0			$-a$				1		* y_3^*
$0 \leq \sigma_4^*$ =	0				$-a$				1	* y_4^*
	*	*	*	*	*	*	*	*	*	
	X_0^1 (= 1) Exogenous	X_1	X_2	X_3	X_4	X_1^*	X_2^*	X_3^*	X_4^*	
		Production activities				0,1 Fixed cost activities				

$W'E'$ (each of which is the analogue of line OY in figure 1(*b*)); this summed line is $E'F'$.

For an aglebraic formulation of the above problem replace each a occurring in the column of a given X_j in table 2 by $1/C_j$ and reduce the fixed-cost vector in the top two rows of the corresponding X_j^* to the vector representing a single fixed-cost incurrence step, where C_j is the capacity limit corresponding to such a step. With this emendation the X_j^* become integer variables that can take on optimal values exceeding unity. The proportionality constraints now become

$$X_j/C_j \leq X_j^*.$$

These constraints, together with the integrality requirement for X_j^*, will lead to the incurrence of an additional fixed-cost step whenever X_j exceeds $C_1, 2C_1, \ldots$, etc. In table 3 an a corresponding to X_j^* and associated with complex w is replaced by $S_{j,w}/C_j$, where $S_{j,w}$ is the scale of activity X_j when complex w is utilized at unit λ weight; fixed costs in the columns of the X_j^* variables are reduced the same way as in table 2.

Table 6

APPROXIMATION 2 NON-CONVEX PROGRAMMING PROBLEM

			Sector 1				Sector 2				
		Min!	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	$0 \geq$	
		π_0	π_A =	π_B =	π_C =	π_D =	π_E =	π_F =	π_G =	π_H =	
Capital	Max! $\sigma_0 =$	350	−142.5	−148.6	−221	−248.4	−109.3	−130	−142.5	−235	* $p_K \equiv y_0$ (=1)
Labour	$0 \leq \sigma_1 =$	2000	−1237.5	−1121.4	−1100	−984.2	−946.4	−705	−837.5	−675	* $p_L \equiv y_1$
		•	•	•	•	•	•	•	•	•	
		X_0 (=1)	λ_A	λ_B	λ_C	λ_D	λ_E	λ_F	λ_G	λ_H	
		Exogenous	0,1 Averaging weights for complexes								

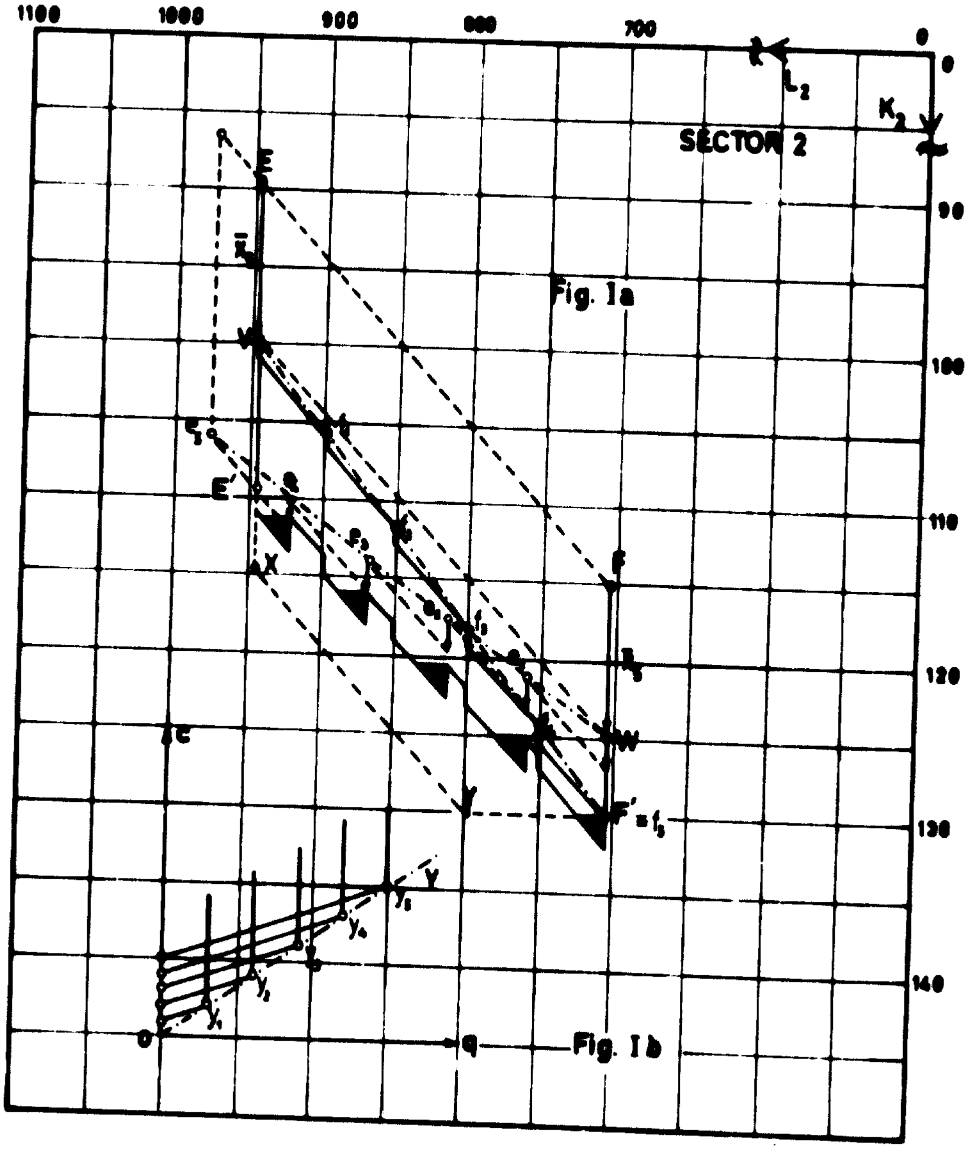

Figure I

STEPWISE INCURRENCE OF FIXED COSTS

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Aerial view of railway yard supplying a coal pier at Norfolk, Virginia, U.S.A.

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